PARTIALLY COMMON VALUES IN AUCTIONS

by

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Abstract

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This research is focused on the partially common (interdependent) values in the three types of the sealed-bid auctions. We introduced the restricted parameter to the value functions that controls the weights which each bidder assigns to his or her signal and the opponent's one. Assuming the existence of the Bayesian-Nash equilibrium in the strictly increasing strategies, we have been looking for it and investigating whether the optimal strategies and the expected seller's revenue are dependent on the introduced parameter. For the auctions where the auctioneer's expected revenue dependents on the parameter, we conducted the revenue comparison analysis.

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Chapter 1

INTRODUCTION

Traditionally all auctions are divided in two extreme groups according to the type of bidders' valuations: private and common value auctions. However, these pure models do not commonly appear in the real world, because people used to evaluate objects in more sophisticated way. Players always have their own valuations about an object, and their own valuations are influenced by others' opinions at the same time. For instance, it is not really correct to exclude the possibility of reselling an object by a winner in the future, and that is the reason why people definitely take into account others' expected valuations of an object when they are bidding.

Nevertheless, theory always simplifies behavior of bidders, there are cases between private and common values which potentially are more precise and definitely worth studying. For instance, Goeree and Offerman (2003) introduced new valuation type in which bidders' signals consist of private and common value components at the same time. Birulin and Izmalkov (2011) studied efficiency properties of the English auction with interdependent values and provided a lot of examples of value functions which represent this type of valuation.

We studied interdependent values. The motivation to consider exactly this type of valuation is that usually people possess different information about the value of the object. Bidders have intensive to consider expectations about others' signals, as it might be profitable for them. We are focused on the special form of value functions in the three types of auctions: first-price sealed-bid auctions, second-price sealed-bid auctions and all-pay sealed-bid auctions.

Two first auction types are chosen for the analysis as they are the most fundamental ones in the auction theory. In the first-price sealed-bid auction all bidders submit their bids simultaneously and no one bidder knows the bids which were submitted by other players. The object goes to the player who submitted the highest bid and the winner pays his or her own bid to the seller/auctioneer. In the second-price sealed-bid auction all bidders submit their bids simultaneously and no one bidder knows the bids which were submitted by other players. The winner is the player who submitted the highest bid, but he or she pays the second highest bid to the seller/auctioneer.

In all-pay sealed-bid auctions as in first-price sealed-bid auctions all bids are submitted simultaneously and the winner is the player who submitted the highest one, but every player should pay his or her bid to the seller. Also, all-pay sealed-bid auctions are interesting as they have a wide range of applications. For instance, they are applicable to political contests as politicians usually spend huge amounts of money during election campaigns in order to attract more voters. It is logically to assume that the more money is spent by the politician, the more is the number of voters that he or she has attracted. So, exactly as in the proposed auction model, every player pays, while there is only one election winner.

We are to investigate what are the equilibrium bidding strategies in the symmetric first-price sealed-bid auction, second-price sealed-bid auction and all-pay sealed-bid auction with two bidders under interdependent (partially common) values, where bidders' signals are correlated, and in the first-price sealed-bid auction and the second-price sealed-bid auction with two or three asymmetric bidders where bidders' signals are independent. Then, we are to answer the question which auction is the best one for the auctioneer by comparing the expected revenues.

Thus, the main goal of the thesis is to study three classical auction types with interdependent values and tell which auction among them is the best one for the seller.

The paper is organized as follows. Chapter 2 overviews some related literature about the classical auctions and the auctions with the extensions as the interdependent valuations and asymmetric bidders. Chapter 3 provides the description of the methodology. Chapter 4 presents the investigation of the existence of the Nash equilibrium strategies in the symmetric auctions and the dependence of the strategies on the parameter presented in the value functions. Chapter 5 describes the existence of the Nash equilibrium strategies in the asymmetric auctions, the dependence of the strategies on the parameter presented in the value functions and the revenue comparison analysis. Chapter 6 concludes the main results.

C h a p t e r 2

LITERATURE REVIEW

This chapter describes the books and papers concentrated on all-pay, first- and second-price sealed bid auction with both traditional (pure private and common values) and non-traditional types of valuations.

A lot of published theoretical researches in the auction theory are based on the same fundamental approaches and methodologies of defining the model and its assumptions and investigating the most important questions as the existence of equilibrium bidding strategies, efficiency of the auctions and what is the auctioneer's expected revenue. Menezes and Monteiro (2005) in their book brought together all the important methodologies and results developed and obtained by Paul Klemperer and Vickrey in their early papers. They showed techniques and methodologies of computing equilibrium bidding strategies in the first- and second-price sealed bid auctions with pure private and common values which we follow. Also, there are described private and common values for bidders' signals of both types: independently distributed and correlated. The authors provided revenue comparison analysis for all the auctions and valuations mentioned above. However, they discuss only classical symmetric models and do not introduce models with asymmetric bidders.

Krishna (2010) provides the theoretical basis needed for studying auctions with asymmetric bidders. The author describes the methodology of studying asymmetric auctions, defining the equilibrium bidding strategies and the auctioneer's expected revenue which we use in our research. Also, Krishna introduces the definition of the interdependent values and discusses an intuition behind them. He provided analysis of the auctions with pure common values from the point of view of the special case of the interdependent values when ex-post valuation of an object for all bidders is the same. In addition, Krishna relaxes the assumption of independently distributed signals and describes generally the case of the first- and second-price sealed-bid auctions with interdependent values under affiliation.

Interdepend values are well studied in English auctions. English auction is open-bid auction, where bids increase until there is no bidder who is willing to top the current maximum bid. For example, Birulin and Izmalkov (2011) studied the English auction with interdependent values. They were focused on the efficiency properties, but also the authors provided a lot of examples of value functions which represent this type of valuation. They introduced several models of the English auction where the value functions of the bidders were symmetric or asymmetric and depended on some positive parameter, as well as symmetric or asymmetric and in-depended of any parameters. However, we are focused on the sealed-bid auctions with the interdependent values.

The second-price sealed-bid auction with interdependent values was explicitly introduced by Osborne (2000). The author derived the symmetric equilibrium bidding strategy function for that auction in the case of two bidders. Bidders' signals are independently and uniformly distributed in that model. The form of the value function of the bidder *i* was considered as $v_i = \alpha t_i + \gamma t_j$, where t_j is the other player's signal and $a \geq \gamma \geq 0$, the author showed that this model includes the cases of the independent pure private values as well as pure common values. The obtained result is $b_i = (\alpha + \gamma)t_i$. That is important that the Nash equilibrium strategies directly depend on a and γ. However, Osborne does not do the revenue analysis and did not show how the seller's expected revenue depend on the values of a and γ. This example is very relevant as our study is done for the similar form of the value functions, but we introduce only one parameter for defining the weights which each bidder assigns to his/her signal and the opponent's one. In addition, we relax the assumption of the independent distribution of the signals for the symmetric auctions. However, we follow the same core assumptions about the distribution of the signals, as

Osborne introduced for the symmetric second-price sealed bid auction, in the models with asymmetric bidders.

Nevertheless, John H. Kagel and Dan Levin (2005) did not study the interdependent values; the asymmetry in common value auctions which were introduced by the authors is interesting to be mentioned. Their idea was to add one advantaged bidder to the original common value auction. The bidder has advantage as his ex-post valuation is slightly higher than the corresponding valuations of all other bidders. As a result, the authors showed that this slight bidder's advantage does not change considerably the seller's expected revenue in the second-price sealed-bid auction, however, has an explosive effect on the auctioneer's expected revenue in English auction.

Another auction model which deviates from the classical ones in terms of value function was introduced by Goeree and Theo Offerman (2003). They had the same motivation as we do for studying the auctions with other than classical value functions. Jacob K. Goeree and Theo Offerman (2003) combined interdependent common and private values signals together and obtained the value function of the absolutely new form. For example, the interdependent values include only private signals. They computed symmetric equilibrium bidding strategies in the first- and second-price sealed-bid and English auctions with these combined valuations. It is important to mention that the 'traditional' types of valuations are included to all the auction models as the special cases. We also constructed the value functions for our research in the way they follow the same property. Goeree and Offerman conducted revenue comparison analysis and discovered that all the studied auctions are revenue equivalent. However, it was mentioned in the paper that that result is very sensitive to the assumption that all the signals are independently distributed. That is why we considered correlated signals for the symmetric auctions in the thesis.

Chapter 3

METHODOLOGY

As defined above we are to study what are the equilibrium bidding strategies in symmetric first-price sealed-bid auction, second-price sealed-bid auction and all-pay sealed-bid auction with two bidders under interdependent (partially common) values where bidders' signals are correlated and conduct revenue comparison analysis. Also, we are to study the asymmetric first-price sealed-bid auction, second-price sealed-bid and all-pay auction with two or three bidders. In this chapter we provide the description of the methodology using as the example the case of the symmetric first-price sealed-bid auction with two bidders.

So, we consider the symmetric auctions where two bidders compete for a single object. Both of them receive private signals about the value of an object (s_1 and s_2). Bidders are assumed to be risk-neutral. It is assumed that the signals are correlated and are drown from the [0,1].

The partially common values (interdependent values) will be presented as:

$$v_1 = \alpha s_1 + (1 - \alpha)s_2$$

$$v_2 = \alpha s_2 + (1 - \alpha)s_1$$

where α ($1 \geq \alpha \geq 0.5$) is the parameter controlling the weights which each bidder assigns to his/her signal and the opponent's one.

We suppose that each bidder follows the same strictly increasing bidding strategy $b(*)$.

For example, the first bidder's expected payoff if the first bidder wins in the first-price sealed-bid auction is:

$$\pi_1 = (E(v_1 \mid s_1, b_2 < b_1) - b_1)P(b_2 < b_1),$$

where the first bidder's signal is s_1, his bid is b_1 and the other player's bid is b_2.

The first bidder with signal s_1 is going to choose the strategy b_1 in order to maximize his expected payoff. We can interpret it as choosing some s, where $b_1 = b(s)$, as it was mentioned above that $b(*)$ is strictly increasing.

The first bidder's expected value of the object is:

$$E(\alpha s_1 + (1-\alpha)s_2 \mid b_2 < b_1) = \alpha s_1 + (1-\alpha)E(s_2 \mid b_2 < b_1).$$

$$E(s_2 \mid v_2 < v_1) = \frac{\int_0^s s_2 f(s_2 \mid s_1) d s_2}{\int_0^s f(s_2 \mid s_1) d s_2},$$

where $f(* \mid s)$ is conditional density function.

$$\pi_1 = \int_0^s (\alpha s_1 + (1-\alpha)s_2 - b_1(s)) f(s_2 \mid s_1) d s_2 .$$

Then, using the first order condition and solving a differential equation we calculate the symmetric equilibrium bidding strategies. It is interesting to see how the equilibrium bidding strategies depend on alpha.

The final step is to compute the auctioneer's expected revenue. The auctioneer's expected revenue is the expected value of the highest bid in the first-price sealed-bid auctions, the expected value of the second highest bid in the second-price sealed-bid auctions, the expected total amount bid in all-pay auctions. The auction with the highest expected revenue is considered the optimal auction for a seller among the studied auctions.

C h a p t e r 4

SYMMETRIC AUCTIONS

4.1. Second-Price Sealed-Bid Auction

We consider the auction where 2 bidders compete for a single object. Both of them receive private signals about the value of an object (s_1 and s_2). Bidders are assumed to be risk-neutral. It is assumed that signals are correlated and are drown from the [0,1].

The partially common values (interdependent values) will be presented as:

$$v_1 = \alpha s_1 + (1-\alpha)s_2$$

$$v_2 = \alpha s_2 + (1-\alpha)s_1$$

where α ($1 \geq \alpha \geq 0.5$) is the parameter controlling the weights which each bidder assigns to his/her signal and the opponent's one.

We suppose that each bidder follows the same strictly increasing bidding strategy $b(*)$.

Let us consider that $b(s_2)$ is the second player's bid, where $b(*)$ is assumed to be continuous and strictly increasing function.

The expected payoff of the first player is:

$$\pi_1 = E[(v_1(s_1,s_2) - b(s_2))I_{b(s)>b(s_2)}|s_1] = E[(v_1(s_1,s_2) - b(s_2))I_{s>s_2}|s_1]$$

$$\pi_1 = \int_0^s (\alpha s_1 + (1-\alpha)s_2 - b(s_2))f(s_2|s_1)ds_2,$$

where $f(*\,|\,s)$ is conditional density function.

The first-order condition:

$$\pi_1' = (\alpha s_1 + (1-\alpha)s - b_2(s))f(s\,|\,s_1),$$

$$(\alpha s_1 + (1-\alpha)s - b(s))f(s\,|\,s_1) = 0,$$

$$\alpha s_1 + (1-\alpha)s - b(s) = 0.$$

$b(s) = \alpha s_1 + (1-\alpha)s$, we can say that $b(s)$ is equal to $\alpha s + (1-\alpha)s$ because the value function is continuous and increasing as the bidding strategy function is.

So, it is optimally to set $s = s_1$.

Thus, the equilibrium bidding strategy is $b^{eq}(s) = s$.

4.2 First-Price Sealed-Bid Auction

We again suppose that each bidder follows the same strictly increasing bidding strategy $b(*)$. The calculation technique follows from Menezes and Monteiro (2005).

The expected payoff of the first bidder is:

$$\pi_1 = \int_0^s (\alpha s_1 + (1-\alpha)s_2 - b(s))f(s_2\,|\,s_1)ds_2,$$

where $b(s)$ is the first player's bid.

$$\pi_1 = \int_0^s (\alpha s_1 + (1-\alpha)s_2) f(s_2 \mid s_1) ds_2 - b(s) F(s \mid s_1)$$

The first-order condition:

$$\pi_1' = (\alpha s_1 + (1-\alpha)s) f(s \mid s_1) - b'(s) F(s \mid s_1) - b(s) f(s \mid s_1)$$

After rearranging the first-order condition is:

$$\pi_1' = (\alpha s_1 + (1-\alpha)s - b(s)) f(s \mid s_1) - b'(s) F(s \mid s_1),$$

$$(\alpha s_1 + (1-\alpha)s - b(s)) f(s \mid s_1) - b'(s) F(s \mid s_1) = 0.$$

We obtained the following differential equation:

$$b'(s) = \frac{(\alpha s_1 + (1-\alpha)s - b(s))}{F(s \mid s_1)} f(s \mid s_1).$$

Let us set $s = s_1$.

$$b'(s_1) = \frac{(\alpha s_1 + (1-\alpha)s_1 - b(s_1))}{F(s_1 \mid s_1)} f(s_1 \mid s_1),$$

$$b'(s_1) + b(s_1) \frac{f(s_1 \mid s_1)}{F(s \mid s_1)} = (\alpha s_1 + (1-\alpha)s_1) \frac{f(s_1 \mid s_1)}{F(s_1 \mid s_1)}.$$

Then, in order to solve this equation, we use the integrating factor method. It is easy to see that integrating factor has to solve $P' = P \frac{f(s_1 \mid s_1)}{F(s_1 \mid s_1)}$.

$P(s) = exp[-\int_s^1 \frac{f(u \mid u)}{F(u \mid u)} du]$ is an integrating factor. Thus,

$$(Pb)'(s_1) = P(s_1) b'(s_1) + P'(s_1) b(s_1).$$

Substituting P' with $P\frac{f(s_1|s_1)}{F(s_1|s_1)}$:

$$(Pb)'(s_1) = P(s_1)b'(s_1) + P(s_1)b(s_1)\frac{f(s_1|s_1)}{F(s_1|s_1)}.$$

Substituting $b'(s_1) + b(s_1)\frac{f(s_1|s_1)}{F(s|s_1)}$ with $(\alpha s_1 + (1-\alpha)s_1)\frac{f(s_1|s_1)}{F(s_1|s_1)}$:

$$(Pb)'(s_1) = P(s_1)(\alpha s_1 + (1-\alpha)s_1)\frac{f(s_1|s_1)}{F(s_1|s_1)}.$$

As $b(0) = 0$ and $P(0) \leq 1$, the limits are 0 and s_1:

$$(Pb)(s_1) = \int_0^{s_1} P(u)(\alpha u + (1-\alpha)u)\frac{f(u|u)}{F(u|u)}du.$$

Representing the above expression in two forms:

$$(Pb)(s_1) = \int_0^{s_1} uP(u)\frac{f(u|u)}{F(u|u)}du \tag{1}$$

$$(Pb)(s_1) = \int_0^{s_1} uP'(u)du \tag{2}$$

From (1) and $P(s) = exp[-\int_s^1 \frac{f(u|u)}{F(u|u)}du]$ we obtain:

$$b(s_1) = \int_0^{s_1} u\frac{f(u|u)}{F(u|u)}exp[-\int_u^{s_1} \frac{f(v|v)}{F(v|v)}dv]du.$$

From (2) by integrating by parts and rearranging we obtain:

$$b(s_1) = \frac{P(s_1)s_1 - \int_0^{s_1} P(u)du}{P(s_1)}.$$

Then, $b(s_1) = s_1 - \frac{\int_0^{s_1} P(u)du}{P(s_1)}$. We can conclude that the equilibrium strategy does not depend on the parameter.

Let us check if it is really an equilibrium:

$$\pi_1' = (\alpha s_1 + (1-\alpha)s - b(s))f(s\,|\,s_1) - b'(s)F(s\,|\,s_1)$$

$$\pi_1' = F(s\,|\,s_1)(\alpha s_1 + (1-\alpha)s - b(s))\frac{f(s\,|\,s_1)}{F(s\,|\,s_1)} - b'(s))$$

If $s_1 > s$:

$$\alpha s_1 + (1-\alpha)s < \alpha s_1 + (1-\alpha)s_1,$$

and, as signals are affiliated, $\frac{f(s\,|\,s)}{F(s\,|\,s)} < \frac{f(s\,|\,s_1)}{F(s\,|\,s_1)}$.

Then,

$$\pi_1' = F(s\,|\,s_1)(\alpha s_1 + (1-\alpha)s - b(s))\frac{f(s\,|\,s_1)}{F(s\,|\,s_1)} - b'(s))$$
$$> F(s\,|\,s_1)(\alpha s + (1-\alpha)s - b(s))\frac{f(s\,|\,s)}{F(s\,|\,s)} - b'(s)) = 0$$

If $s_1 < s$, then analogically we get that $\pi_1' < 0$. That is why in order to maximize the payoff we choose $s = s_1$.

4.3 All-pay auction

We assume that players follow the same strictly increasing bidding strategy $b(*)$. The first bidder's signal is s_1 and the bid is $b(s)$. So, the expected payoff is:

$$\pi_1 = \int_0^s (\alpha s_1 + (1-\alpha)s_2) f(s_2 \mid s_1) ds_2 - b(s).$$

The first-order condition:

$$(\alpha s_1 + (1-\alpha)s) f(s \mid s_1) - b'(s) = 0.$$

Let us set $s = s_1$ and rearrange:

$$b'(s_1) = (\alpha s_1 + (1-\alpha)s_1) f(s_1 \mid s_1),$$

$$b(s_1) = \int_0^{s_1} s f(s \mid s_1) ds.$$

Let us check if it is really an equilibrium:

$$\pi_1' = (\alpha s_1 + (1-\alpha)s) f(s \mid s_1) - b'(s).$$

If $s > s_1$, then $\pi_1' < 0$. Also, $\pi_1' > 0$, when $s < s_1$. That is why the first bidder's expected payoff is maximized only if $s = s_1$

From all above we can conclude that the equilibrium strategies in the discussed auctions with the proposed value functions do not depend on the parameter α.

Chapter 5

ASYMMETRIC AUCTIONS

5.1 Second-Price Sealed-Bid Auction with asymmetric bidders

Let us consider the auction where two bidders compete for a single object. Both of them receive private signals about the value of an object (s_1 and s_2). Bidders are assumed to be risk-neutral. It is assumed that signals are independently and uniformly distributed on the [0,1].

The first bidder's valuation depends not only on his own type, but the other bidder's type as well. So, this is an auction with interdependent valuations. The second bidder has independent private values.

$$v_1 = \alpha s_1 + (1-\alpha)s_2$$

$$v_2 = s_2$$

We assume that the second bidder bids his signal, as it is known from the previous literature that for this player it is his weakly dominant strategy. Also, we assume that the first bidder follows the strictly increasing bidding strategy $b(*)$. Let us suppose for simplicity that this is the strategy of the form $b(s) = \beta s$ (where β is some positive constant). Now we can compute the first bidder's expected payoff if he bids $b = \beta s$, given that the second bidder bid his signal.

Then,

$$\pi_1 = \int_0^{b/\beta} (\alpha s_1 + (1-\alpha)s_2 - s_2))ds_2 = \alpha \int_0^{b/\beta} (s_1 - s_2)ds_2,$$

$$\pi_1 = \frac{\alpha b s_1}{\beta} - \frac{\alpha b^2}{2\beta^2}.$$

The first-order condition:

$$\frac{\alpha b' s_1}{\beta} - \frac{\alpha b' b}{\beta^2} = 0$$

So, we see that the equilibrium strategy is $b = \beta s_1$.

$$\pi_2 = \int_0^s (s_2 - \beta s_1) d s_1 = s_2 s - \frac{\beta s^2}{2} \ ,$$

$$\pi_2' = s_2 - \beta s \ .$$

As we know that the weakly dominant strategy for the second bidder is to bid his signal, we set $s = s_2$. Then, $\beta = 1$ and it does not depend on the parameter α.

Let us consider the same auction with three bidders. The first and the second bidder have interdependent values, while the third bidder has private values:

$$v_1 = \alpha s_1 + (1 - \alpha) s_2$$

$$v_2 = \alpha s_2 + (1 - \alpha) s_1$$

$$v_3 = s_3$$

We use the same strategy as in the above case. So, we assume that the third bidder bids his signal and that the first and the second players follow the strictly increasing bidding strategy $b(*)$. We suppose again for simplicity the linear solution of the form $b(s) = \beta s$ (where β is some positive constant) for the first and the second bidders.

So, the expected payoff of the first player if he bids some b, given that the third player bids his signal and the second player bids $b(s_2) = \beta s_2$, is:

$$\pi_1 = \int_0^{b/\beta} \left(\int_0^{\beta s_2} (\alpha s_1 + (1-\alpha)s_2 - \beta s_2) d s_3 + \int_{\beta s_2}^{b} (\alpha s_1 + (1-\alpha)s_2 - s_3) d s_3 \right) d s_2,$$

where $\int_0^{\beta s_2} (\alpha s_1 + (1-\alpha)s_2 - \beta s_2) d s_3$ is for the case when the second bidder's bid is the second highest bid and $\int_{\beta s_2}^{b} (\alpha s_1 + (1-\alpha)s_2 - s_3) d s_3$ is for the case when the third bidder's bid is the second highest bid.

After integrating we get:

$$\pi_1 = \frac{\alpha b^2 s_1}{\beta} - \frac{b^3}{3\beta^2} + \frac{(1-\alpha)b^3}{2\beta^2} - \frac{b^3}{2\beta} + \frac{b^3}{6\beta}.$$

The first-order condition:

$$\frac{2\alpha s_1 b b'}{\beta} - \frac{b^2 b'}{\beta^2} + \frac{3(1-\alpha)b^2 b'}{2\beta^2} - \frac{3b^2 b'}{2\beta} + \frac{b^2 b'}{2\beta} = 0,$$

$$b = \frac{2\alpha\beta s_1}{\alpha + 2\beta - 1}.$$

Notice that if we consider $\alpha = 1$, then we obtain the auction with pure private values and the equilibrium bidding strategy above will be equal to $b = s_1$. It is consistent with the theory, so the computed equilibrium strategy is correct.

As the first and the second players are symmetric, we can calculate β as:

$$\beta = \frac{2\alpha\beta}{\alpha + 2\beta - 1},$$

$$\beta = \frac{1}{2}\alpha + \frac{1}{2}.$$

We can conclude that the parameter α has positive effect on the bidding strategies.

Our next step is to investigate how the parameter α influences the seller's revenue.

5.2 First-Price Sealed-Bid Auction with asymmetric bidders

Let us consider the auction where two bidders compete for a single object. Both of them receive private signals about the value of an object (s_1 and s_2). Bidders are assumed to be risk-neutral. It is assumed that signals are independently and uniformly distributed on the [0,1].

The first bidder's valuation depends not only on his own type, but the other bidder's type as well. So, this is an auction with interdependent valuations. The second bidder has independent private values

$$v_1 = \alpha s_1 + (1 - \alpha)s_2$$

$$v_2 = s_2$$

We assume that the second bidder bids his signal, as it is known from the previous literature that for the player it is optimally to bid his/her signal under the case of the independent private values. Also, we assume that the first player follows the strictly increasing bidding strategy $b(*)$.

Let us suppose for simplicity the linear solution of the form $b(s) = \beta s$ (where β is some positive constant) for the first bidder. Then, we build the second bidder's expected payoff as the product of the probability of his winning and his expected payoff in that case:

$$\pi_2 = P(b_2 > b_1)(E[v_2 \,|\, s_2, b_2 > b_1] - b_2).$$

Substituting b_1 with βs_1 and computing the probability and the expected value:

$$\pi_2 = \frac{b_2}{\beta}(s_2 - b_2).$$

The first-order condition:

$$\frac{b_2'}{\beta}(s_2 - 2b_2) = 0.$$

So, given that the first bidder strategy is linear, the optimal strategy for the second bidder is $b_2 = \dfrac{1}{2}s_2$.

Now let us find the optimal strategy of the first bidder, given that the second bidder bids half of his valuation

We build the first bidder's expected payoff as the product of the probability of his winning and his expected payoff in that case:

$$\pi_1 = P(b_1 > b_2)(E[v_1 \,|\, s_1, b_1 > b_2] - b_1).$$

Substituting b_2, v_1 with $\dfrac{1}{2}s_2$ and $\alpha s_1 + (1-\alpha)s_2$, respectively, we got:

$$\pi_1 = P(b_1 > \frac{1}{2}s_2)(E[\alpha s_1 + (1-\alpha)s_2 \,|\, s_1, b_1 > \frac{1}{2}s_2] - b_1).$$

Rearranging and computing the expected value of s_2 and the probability:

$$\pi_1 = 2\alpha b_1(s_1 - b_1).$$

The first-order condition with respect to b_1:

$$2\alpha b_1'(s_1 - 2b_1) = 0.$$

The solution is $b_1 = \frac{1}{2}s_1$.

So, we can conclude that the equilibrium bidding strategies in that auction do not depend on α. Also, that means the expected revenue of the seller does not depend on it.

Let us consider the auction where 3 bidders compete for a single object. All of them receive private signals about the value of an object (s_1, s_2 and s_3). Bidders are assumed to be risk-neutral. It is assumed that signals are independently and uniformly distributed on the [0,1].

The first and the second bidders have interdependent values, while the third bidder has private values:

$$v_1 = \alpha s_1 + (1-\alpha)s_2$$

$$v_2 = \alpha s_2 + (1-\alpha)s_1$$

$$v_3 = s_3$$

We assume that the first and the second players follow the same strictly increasing bidding strategy $b(*)$, as their value functions are symmetric. Also, we suppose for simplicity that their strategy is of the form $b_i(s_i) = \beta s_i$ (where β is some positive constant and s_i is signal, $i = 1,2$).

Let us build the third bidder's expected payoff as the product of the probability of his winning and his expected payoff in that case:

$$\pi_3 = P(b_3 > b_1, b_2)(E[v_3 \,|\, s_3, b_3 > b_1, b_2] - b_3).$$

Substituting b_1, b_2 with $\beta s_1$and βs_2, respectively, and computing the expected value, we got:

$$\pi_3 = P(b_3 > \beta s_1, \beta s_2)(s_3 - b_3),$$

$$\pi_3 = P(\frac{b_3}{\beta} > s_1, \frac{b_3}{\beta} > s_2)(s_3 - b_3).$$

After calculating the probabilities and opening the brackets, we obtained the final expression for the third bidder's payoff:

$$\pi_3 = \frac{b_3^2 s_3}{\beta^2} - \frac{b_3^3}{\beta^2}.$$

The first-order condition with respect to b_3:

$$\frac{2b_3' b_3 s_3}{\beta^2} - \frac{3b_3^2 b_3'}{\beta^2} = 0.$$

The solution is $b_3 = \frac{2}{3} s_3$.

It is interesting to notice that the obtained above bidding strategy coincides with the one in the symmetric first-price sealed-bid auction with pure private values for three bidders. The general formula for the strategy is $b_i = \frac{n-1}{n} s_i$, where n is the number of bidders.

Our next step is to find the first bidder's response assuming that the second player follows a linear strategy $b_2(s_2) = \beta s_2$ (where β is some positive constant) and that the third player uses the strategy $b_3(s_3) = \frac{2}{3}s_3$

Let us build the first bidder's expected payoff as the product of the probability of his winning and his expected payoff in that case:

$$\pi_1 = P(b_1 > b_2, b_3)(E[v_1 | s_1, b_1 > b_2, b_3] - b_1)$$

Substituting b_2, b_3, v_1 with βs_2, $\frac{2}{3}s_3$ and $\alpha s_1 + (1-\alpha)s_2$, respectively, we got:

$$\pi_1 = P(b_1 > \beta s_2, \frac{2}{3}s_3)(E[\alpha s_1 + (1-\alpha)s_2 | s_1, b_1 > \beta s_2] - b_1).$$

Rearranging and computing the expected value of s_2:

$$\pi_1 = P(\frac{b_1}{\beta} > s_2, \frac{3}{2}b_1 > s_3)(\alpha s_1 + (1-\alpha)E[s_2 | \frac{b_1}{\beta} > s_2] - b_1)$$

$$= \frac{3b_1^2}{2\beta}(\alpha s_1 + (1-\alpha)E[s_2 | \frac{b_1}{\beta} > s_2] - b_1)$$

$$= \frac{3b_1^2}{2\beta}(\alpha s_1 + (1-\alpha)\frac{b_1}{2\beta} - b_1)\ .$$

The first-order condition:

$$b_1'((\frac{3b_1}{\beta}(\alpha s_1 + (1-\alpha)\frac{b_1}{2\beta} - b_1) + \frac{3b_1^2}{2\beta}(\frac{1-\alpha}{2\beta} - 1)) = 0$$

The solution is $b_1 = \frac{4\beta a s_1}{6\beta + 3a - 3}$.

As the first and the second bidders had symmetric value functions, the second bidder's strategy is assumed to be the same.

Notice that if we consider $\alpha = 1$, then we obtain the symmetric auction with pure private values and the equilibrium bidding strategy above will be equal to $b_1 = \frac{2}{3}s_1$. It is consistent with the theory, so the computed equilibrium strategy is correct.

As the first and the second player are symmetric, we can calculate β as:

$$\beta = \frac{4\beta a}{6\beta + 3a - 3},$$

$$\beta = \frac{1}{6}\alpha + \frac{1}{2}.$$

We can conclude that the parameter α has positive effect on the bidding strategies.

As in the previous case, our next step is to investigate how the parameter α will influence the seller's revenue.

5.3 All-pay Auction with asymmetric bidders

Let us consider the auction where 3 bidders compete for a single object. All of them receive private signals about the value of an object (s_1, s_2 and s_3). Bidders are assumed to be risk-neutral. It is assumed that signals are independently and uniformly distributed on the [0,1].

The first and the second bidders have interdependent values, while the third bidder has private values:

$$v_1 = \alpha s_1 + (1 - \alpha)s_2$$

$$v_2 = \alpha s_2 + (1 - \alpha)s_1$$

$$v_3 = s_3$$

We assume that the first and the second players follow the same strictly increasing bidding strategy $b(*)$, as their value functions are symmetric. Also, we suppose for simplicity that their strategy is of the form $b_i(s_i) = \beta s_i^3$ (where β is some positive constant and s_i is signal, $i = 1,2$). As in the symmetric all-pay auction with pure private values, uniformly distributed signals and n bidders the equilibrium bidding strategy is $b_i = s^n - \int_0^s s'^{n-1} ds'$, that is why we assume that the bidding strategies will contain the cube of the player's signal in our case also.

Let us build the third bidder's expected payoff as the product of the probability of his winning and his expected payoff in that case:

$$\pi_3 = P(b_3 > b_1, b_2)(E[v_3 \,|\, s_3, b_3 > b_1, b_2] - b_3) - (1 - P(b_3 > b_1, b_2))b_3.$$

Substituting b_1, b_2, v_3 with βs_1^3, βs_2^3 and s_3, respectively, and computing the expected value, we got:

$$\pi_3 = P(b_3 > \beta s_1^3, \beta s_2^3)(s_3 - b_3) - (1 - P(b_3 > \beta s_1^3, \beta s_2^3)b_3.$$

After calculating the probabilities, we obtained the final expression for the third bidder's payoff:

$$\pi_3 = \sqrt[3]{\frac{b_3^2}{\beta^2}} s_3 - b_3.$$

The first-order condition:

$$b_3'(\frac{2s_3}{3\beta^2}(\frac{b_3^2}{\beta^2})^{-\frac{2}{3}}b_3 - 1) = 0.$$

The solution is $b_3 = \frac{8}{27\beta^2}s_3^3$.

Our next step is to find the first bidder's response assuming that the second player follows a strategy $b_2(s_2) = \beta s_2^3$ (where β is some positive constant) and that the third player uses the strategy $b_3(s_3) = \frac{8}{27\beta^2}s_3^3$.

Let us build the first bidder's expected payoff as the product of the probability of his winning and his expected payoff in that case:

$$\pi_1 = P(b_1 > b_2, b_3)(E[v_1 \,|\, s_1, b_1 > b_2, b_3] - b_1) - (1 - P(b_1 > b_2, b_3))b_1,$$

$$\pi_1 = P(b_1 > \beta s_2^3, \frac{8}{27\beta^2}s_3^3)(E[\alpha s_1 + (1-\alpha)s_2 \,|\, s_1, b_1 > \beta s_2^3] - b_1) - (1 - P(b_1 > \beta s_2^3, \frac{8}{27\beta^2}s_3^3))b_1$$

After calculating the probabilities, we obtained the final expression for the first bidder's payoff:

$$\pi_1 = \sqrt[3]{\frac{27b_1^2\beta}{8}}(\alpha s_1 + \frac{(1-\alpha)}{2}\sqrt[3]{\frac{b_1}{\beta}} - b_1).$$

Taking the first-order conditions we obtain:

$$(\frac{\beta(\frac{1}{2}(1-\alpha)\sqrt[3]{\frac{b_1}{\beta}} + \alpha s_1 - b_1}{2(\beta b_1)^{\frac{2}{3}}} + \frac{3}{2}\sqrt[3]{\beta b_1}(\frac{1-\alpha}{6\beta(\frac{b_1}{\beta})^{\frac{2}{3}}} - 1))b_1' = 0.$$

It can be seen from above that there is a relationship between α, β and b_1. However, the equation cannot be solved analytically. That is why, this case we leave as an opportunity for the future researches.

5.4 Revenue Comparison

We calculated asymmetric equilibrium strategies for two auctions. In this chapter we want calculate and compare seller's expected revenues for those models. Let us start from the second-price sealed bid auction with 3 asymmetric bidders.

At the beginning, we have to calculate the distribution of the price paid to the seller by the winner as the distribution of the second-highest bid.

That price is districted on the set $\{(\frac{1}{2}a+\frac{1}{2})s_1, (\frac{1}{2}a+\frac{1}{2})s_2, s_3\}$.

So, there are two possible events:

1) Two of $\{(\frac{1}{2}a+\frac{1}{2})s_1, (\frac{1}{2}a+\frac{1}{2})s_2, s_3\}$ are less than price (we will dente it as p) and one is larger than p;

2) All elements of $\{(\frac{1}{2}a+\frac{1}{2})s_1, (\frac{1}{2}a+\frac{1}{2})s_2, s_3\}$ are equal or less than p.

The resulted distribution function is $F(p) = \dfrac{p^2 + 2(\frac{1}{2}a+\frac{1}{2})p^2 - 2p^3}{(\frac{1}{2}a+\frac{1}{2})^2}$ for any $p \leq \frac{1}{2}a + \frac{1}{2}$.

The expected revenue of the seller is the expected value of the price that is going to be paid by the winner of the auction:

$$E(Revenue) = \int_0^{\frac{1}{2}a+\frac{1}{2}} p\,dF(p).$$

$$E(Revenue) = \int_0^{\frac{1}{2}a+\frac{1}{2}} \frac{1}{(\frac{1}{2}a+\frac{1}{2})^2}(2p^2 + 4(\frac{1}{2}a + \frac{1}{2})p^2 - 6p^3)dp$$

$$= -\frac{1}{36}a^2 + \frac{10}{36}a + \frac{11}{36}.$$

Now let us consider the first-price sealed bid auction with 3 asymmetric bidders.

We need to calculate the distribution of the price paid to the seller by the winner as the distribution of the highest bid.

That price is districted on the set $\{(\frac{1}{6}a + \frac{1}{2})s_1, (\frac{1}{6}a + \frac{1}{2})s_2, \frac{2}{3}s_3\}$.

So, the event that we are looking for occurs when all elements of the set $\{(\frac{1}{6}a + \frac{1}{2})s_1, (\frac{1}{6}a + \frac{1}{2})s_2, \frac{2}{3}s_3\}$ are equal or less than p.

The resulted distribution function is $F(p) = \frac{3p^3}{2(\frac{1}{6}a + \frac{1}{2})^2}$ for any $p \le \frac{1}{6}a + \frac{1}{2}$.

The expected revenue of the seller is the expected value of the price that is going to be paid by the winner of the auction:

$$E(Revenue) = \int_{0}^{\frac{1}{6}a+\frac{1}{2}} p dF(p),$$

$$E(Revenue) = \int_{0}^{\frac{1}{6}a+\frac{1}{2}} \frac{9p^3}{2(\frac{1}{6}a+\frac{1}{2})^2} dp = \frac{1}{32}\alpha^2 + \frac{6}{32}\alpha + \frac{9}{32}.$$

The expected revenue of the seller in the second-price sealed-bid auction is represented by the blue line and the expected revenue of the seller in the first-price sealed-bid auction is represented by the red line in Fig. 1.

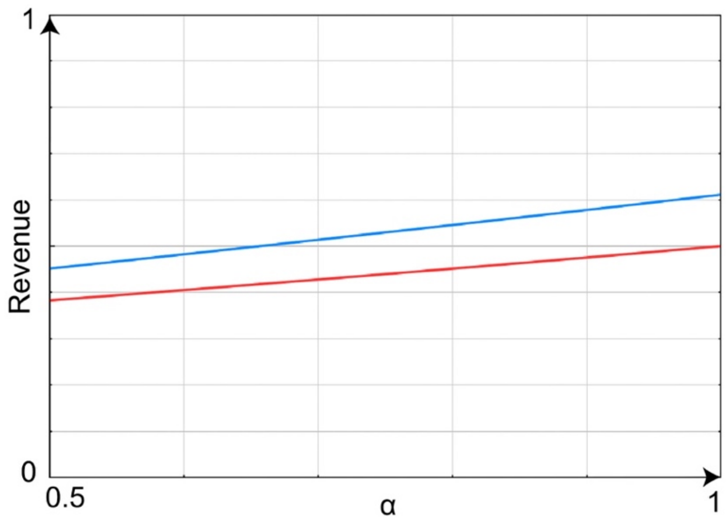

Figure 1. The expected revenues of the seller

We can conclude that the larger is α, the larger is expected revenue of the seller. Recalling the meaning of the parameter, the larger is the weight assigned by the bidder to his own signal (the closer the value functions are to independent pure private value form), the more the auctioneer will earn. Also, we can see that under proposed auction models, conducting the second-price sealed-bid auction is more profitable for the auctioneer than conducting the first-price sealed-bid auction.

Chapter 6

CONCLUSIONS

The traditional auction theory always simplifies the reality. Classical auction models are designed under a big number of assumptions as symmetry of the bidders, identically and independently disturbed value signals, risk neutrality of the bidders, pure common or pure private valuations. However, these models do not have a lot in common with the real ones.

In this thesis we studied classical auctions with interdependent value functions with the parameter. We considered models with 2 symmetric bidders and correlated signals (the first- and the second-price sealed-bid auctions and all-pay auction) and with 2 or 3 asymmetric bidders and independent signals (the first- and the second-price sealed-bid auctions and all-pay auction).

We observed that in the symmetric auctions the value of the parameter presented in the value functions does not have any influence on the equilibrium bidding strategies, as in the maximization of the player's payoff it was cancelled out. All of that, in turn, means that the seller's revenue does not depend on the parameter either. It is possible to tell that under proposed value functions the studied models will "converge" to the case of the pure private values. The same result has been obtained for the first- and the second-price asymmetric auctions with two players. We did not obtain the optimal strategies in the asymmetric all-pay auction as the maximization of the first bidder's payoff cannot be solved analytically. We leave this case for the future studying.

However, in asymmetric auctions with three players the parameter has a positive effect on the bidding strategies and the seller's revenue. So, the larger is the weight assigned by the bidder to his own signal, the larger is submitted bid and the seller's expected revenue.

Also, comparing the first- and the second-price sealed-bid auctions, it is more profitable for sellers to choose the second-price auction, as under any value of the parameter, that auction brings greater expected revenue than the first-price auction.

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