UH/KSE Fellowship Exam – 2019

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1 Differential equations

Solve the equation below for $y \geq 0$, find the steady states (if any), identify their stability. Which value of $y = y^*$ corresponds to the maximum growth rate of y? (Note: A dot over a variable means the first derivative with respect to its argument, time. Hint: steady states correspond to $\dot{y}$=0)

$$\dot{y} = \sqrt{y} - \frac{1}{2}y + 1 \tag{1}$$

Solution:

We need to find those y_i which turn the right-hand side of the above equation into zero. Therefore, we need to solve $\sqrt{y}+1-\frac{1}{2}y = 0$ and find the corresponding y_1 and y_2. After solving this quadratic equation, we get: $y_1 = (1+\sqrt{3})^2$ and $y_2 = (1-\sqrt{3})^2$. A steady-state is stable when $(\dot{y})' < 0$. The second root y_2 satisfies this condition.

The function is concave, $y(0) = 1$, and it reaches maximum at $y^{**} : (\dot{y})' = 0$. $(\dot{y})' = \frac{1}{2\sqrt{y}} - \frac{1}{2}$, therefore, $y^{**} = 1$.

2 Optimization

Find the relative extrema for the following function by (1) finding the critical values(s) and (2) determining if at the critical value(s) the function is at a relative maximum or minimum: $f(x) = 3x^3 - 36x^2 + 135x - 13$.

Solution:

If you find the FOC ($f' = 0$) and simplify by 9, you would get the following quadratic equation $x^2 - 8x + 15 = 0$ with roots equal to 3 and 5. Those are the critical points. To test whether they correspond to a maximum or a minimum, need to find $f'' = 0$ which is $2x - 8$. For $x = 3$, it's negative (thus, a maximum) and for $x = 5$ it's positive (thus, a minimum).

3 Implicit function theorem

$M(\cdot)$ is the marginal utility function with the following properties: $M(\cdot) > 0$ and $M'(\cdot) < 0$. You are given the information that in the equilibrium of the so-called habit formation model:

$$M(c) - M(w - c - \gamma c) \cdot \{1 + \gamma\} = 0 \tag{2}$$

where c is consumption, w is the level of wages which you can assume to be exogenous and treat as a parameter, and γ is the measure of "habit strength." How equilibrium consumption c depends on γ, in other words, what is $\partial c/\partial\gamma$? Can you comment on its sign?

Solution:

Let's first rewrite the equation: $M(c) - M(w - c\{1+\gamma\})\{1+\gamma\} = 0$. Now, let's differentiate the equation wrt γ while keeping in mind that $c = c(\gamma)$: $M'(c)\frac{\partial c}{\partial\gamma} - M'(w - c\{1+\gamma\})[-\frac{\partial c}{\partial\gamma}\{1+\gamma\} - c]\{1+\gamma\} - M(w - c\{1+\gamma\}) = 0$. Don't forget that whatever is inside the parenthesis – is just an argument, in other words – we evaluate M in that point. Let's denote $w - c\{1+\gamma\}$ as $\tilde{c}$. Thus, $M'(c)\frac{\partial c}{\partial\gamma} + M'(\tilde{c})\{1+\gamma\}^2\frac{\partial c}{\partial\gamma} + c\{1+\gamma\}M'(\tilde{c}) - M(\tilde{c}) = 0$. Express c'_γ: $\frac{\partial c}{\partial\gamma} = \frac{M(\tilde{c}) - c\{1+\gamma\}M'(\tilde{c})}{M'(c) + M'(\tilde{c})\{1+\gamma^2\}} < 0$ as the numerator is negative and the denominator is positive.

4 Optimization with Constraints

Adopted from: Graduate macroeconomics I, Dietrich Vollrath, Problem B.1.12, page 113. Consider a one-period model in which individuals make a labor-leisure decision. Individuals have one unit of time (time budget). Utility $U(n, C) = \ln n + \frac{C^{1-\sigma} - 1}{1-\sigma}$ where n is share of time spent on leisure and C is consumption level and $\sigma > 0$. Income is earned in the form of wage w, which is taken as given. They live only one period (a static problem). They earn $(1-n)w$ and consume all their earnings. Define labor supply l^* as $l^* = (1 - n^*)$. Show how the effect of the wage on labor supply $l^* = (1 - n^*)$ depends on the value of σ.

Solution:

We know that $c = (1-n)w$. Thus, we can plug it into $U(n, c)$ to obtain $U = U(n)$ only. Let's do it: $U(n) = \ln n + \frac{[(1-n)w]^{1-\sigma} - 1}{1-\sigma}$. Now we need to optimize with respect to n by equating the derivative of $U'(n)$ to zero. $\frac{\partial U}{\partial n} = \frac{1}{n} + [(1-n)w]^{-\sigma}(-w) = \frac{1}{n} - \frac{w}{[(1-n)w]^{\sigma}} = 0$. Therefore, the final equation we are interested in is $n = (1-n)^{\sigma}w^{\sigma-1}$. Now assume that $n = n(w)$ and differentiate both sides with respect to w: $\frac{\partial n}{\partial w} = [(1-n)w]^{\sigma-1}\{\frac{1-n}{w}(\sigma - 1) - \sigma\frac{\partial n}{\partial w}\}$. Express $\frac{\partial n}{\partial w}$ to obtain that $\frac{\partial n}{\partial w} = \frac{(1-n)(\sigma-1)}{\{[(1-n)w]^{1-\sigma} + \sigma\}w}$. As $0 < n < 1$, the sign of the derivative depends entirely on the sign of $(1 - \sigma)$.