BIDDING STRATEGIES IN PROCUREMENT-RELATED TWO-STAGE AUCTIONS

by

Yaroslav Kheilyk

A thesis submitted in partial fulfillment of
the requirements for the degree of

MA in Economic Analysis

.

Kyiv School of Economics

2017

Thesis Supervisor: ______________________ Professor Pavlo Prokopovych

Approved by ____
Head of the KSE Defense Committee, Professor Tymofiy Mylovanov

____

____

____

Date ______________________________

Kyiv School of Economics

Abstract

BIDDING STRATEGIES IN PROCUREMENT-RELATED
TWO-STAGE AUCTIONS

by Kheilyk Yaroslav

Thesis Supervisor: Professor Prokopovych Pavlo

The focus of this research is on examining the model of the two-stage auction where the one object is sold on. In the paper, bidders' strategies are derived for the first and second rounds in the case of two players depending on players' type. Bidding strategies for the first round are compared to equilibrium bidder's strategies in the standard first-price sealed-bid auction, and the auctioneer's expected revenue is contrasted with the auctioneer's revenue in the first-price sealed-bid auction.

To all Ukrainian soldiers, officers,
and volunteers who have defended
and continue to defend our country

ACKNOWLEDGMENTS

The author wishes to express exceptional gratitude to his thesis advisor, Professor Pavlo Prokopovych, for supervising of this research, overall guidance, spreading knowledge, important corrections, extraordinary ideas, support and understanding in the most difficult situations.

The author is appreciative to Volodymyr Vakhitov, Maksym Obrizan, Olga Kupets, Hanna Vakhitova and all other professors who were participating in our Research Workshops for many helpful comments on my paper. In addition, the author is thankful to all KSE professors and researchers, who are always ready to help and give valuable consultations at any time. Special thanks to goes to Professor Tymofiy Mylovanov who inspired the author on this research and to Natalia Kalinina who gave important comments how to write in English correctly.

In addition, the author would like to thank all my classmates and first-year students for their beliefs in his ability to finish this theoretical research and spending time on discussions about his issues with research.

The author is especially appreciative to his family: parents, older and younger brothers who provided unprecedented support and believed in the success of the author without alternative.

TABLE OF CONTENTS

GLOSSARY

Sealed-bid auction. The auction where all buyers submit their bids simultaneously, and no buyer knows the offers made by the other buyers.

First-price sealed-bid auction. The sealed-bid auction where the bidder, who offer the highest bet, wins the object and pays his bet.

Two-stage (two-round) auction. The auction consists of two sequential auctions. In the offered model in the paper, the first one is the first-price sealed-bid auction and the second one is the auction with ordered moving players.

Reserve price. The minimum price that the auctioneer is willing to sell the item. In the offered model, the reserve price is set up for second round as the maximum price proposed at first round.

"Aggressive" strategy. The strategy of the player with lower valuation at the second round when he bids the bet which equals to the maximum of his valuation of the object.

"Peaceful" strategy. The strategy of the player with lower valuation at the second round when he bids the bet which equals to the maximum of his valuation of the reserve price.

$\mathbf{b_j^i}$ - the bet of the j-s player at the i-s round

$\mathbf{v_i}$ - the valuation of the i-s bidder

Chapter 1

INTRODUCTION

There are two main characteristics of auction mechanism, defining which auction is better and which is worse. It is an expected revenue of auctioneer and efficiency of the auction. There are a lot of well-studied auctions with theoretically calculated expected auctioneer's revenue: as an open-bid (English auction, Dutch auction) or sealed-bid (first-price sealed-bid auction, Vickrey auction). But nevertheless, auctioneers try to maximize their payoff by using different approaches. For example, reserve price could be applied to improving the profit. Another approach is to introduce the fixed fee for taking part in the auction. Even more, there is a more extremal model when all bidders ought to pay their bid regardless of the fact whether a bidder wins or not. Such type of auction is called all-pay auction.

All these approaches have both advantages and disadvantages. For example, auctions with exogenously defined reserved price could increase the expected revenue of the auction, but at the same time, they increase the probability that the object will not be sold.

One interesting approach, which is commonly used for selling several objects, is a sequential auction. It is the auction that consists of some finite number of rounds (stages), at every round of which there are several objects to be sold (more often one object per one round). There are a lot of research works, which look into different issues of this type of auctions. Milgrom and Weber (Milgrom, et al., 1982) were the first to derive the equilibrium in the sequential auction. The impact of risk-aversion on the bidder's strategies was examined by McAfee and Vincent

(McAfee, et al., 1993). But these researchers studied the case when bidders are single-unit demand. Bret Katzman (Katzman, 1999), in his paper, found symmetric equilibria of the two-stage and two-goods auction when bidders are two-unit demanded.

Nevertheless, there is a lack of research about a multi-round auction where only one object is sold. Although there are some models using a multi-stage auction for describing labor market competition (Virag, 2006) or all-pay two-stage auction for developing the model of the lobby (Hirata, 2014), still there is a lack of studies about the multi-stage auction. The importance of learning this type of auction is very actual now for Ukraine. The Ukrainian government applies the auction mechanism in different fields: procurement (auction in electronic system "Prozorro"), selling of the arrested properties (electronic system "SETAM"). It's especially important to understand the strategy of bidders, which they will apply in a multi-round auction for the procurement system "Prozorro", which uses three-stage auction with a previously defined reserve price. Currently, there are no theoretical researches which could answer this question. But at the beginning of 2017, Swedish economist Giancarlo Spagnolo (Stepaniuk, 2017) conducted the experiment for defining which auction (first-price sealed-bid auction, two-period auction – simplification of "Prozorro" auction and English auction) brings the higher expected revenue for the auctioneer. Results of experiment show that the highest revenue for auctioneer brings first-price sealed-bid auction, second order is two-round auction and the last one is the English auction. These the results a little bit confused the expectations of Spagnolo, because according to his view first-price sealed-bid auction and two-period auction should have the same expected auctioneer's revenue.

In this research, we try to understand how the behavior of bidders will be changing if we apply the concept of several stages for selling one object in case of two-stage auction with two bidders. It is not intuitively understandable whether competitors will use more aggressive strategy comparing to standard first-price sealed-bid auction. One of the main questions of this study is that if the equilibrium of this auction will be the same as for the first-price sealed-bid auction. If it's not true, what perfect equilibrium could be in this auction and whether this equilibrium will be symmetric. In addition, our goal is to estimate the expected revenue of auctioneer and compare it with the standard first-price sealed-bid auction case. The interesting aspect of this study is whether could give theoretical explanations of results of the experiment of professor Spagnolo.

Chapter 2

LITERATURE REVIEW

There are a lot of papers in which different authors consider the mechanism of the auction for different purposes (the efficiency distribution of resources or maximization of revenue), under different assumptions about bidder's valuations, riskiness, incompleteness or imperfection of information set and using different approaches for finding equilibrium. One of the most classical paper about the auction is the article by Milgrom and Weber (Milgrom, et al., 1982) "A Theory of Auctions and Competitive Bidding". In this paper, authors consider a huge range of topics: the first and second-price sealed-bid auction, English auction, revenue and equivalence conditions for different auctions. The paper shows how risk-awareness of bidders influences on their strategy and consequently on the revenue of auctioneer. The impact of the reserve price and entry fee was also covered in the research.

A lot of papers pay attention to the auction efficiency. Birulin (Birulin, 2003) studies the English auction where only one object is sold. In this case, bidders have a different information about the object. He sets up the model where each bidder receives signals about the bid and these signals are jointly distributed with some density function. For solving this auction, the author proposes the own concept of equilibria (so-called ex-post equilibria). He proves that the proposed auction has an infinitely many ex-post equilibria. The ex-post equilibrium is defined as Bayesian-Nash equilibrium but such which guarantees that even the revealing of the information cannot be cause for changing the bidder's strategy. Another case of efficiency considered by Burguet and Sakovics (Burguet, et al., 1999) considers the competition between auctioneers. Authors study the competition between two owners of identical goods, who wish to sell them to a pool of potential buyers. They show that this game has at least one equilibrium, but, since reserve prices don't converge to zero, all equilibria are inefficient.

The very interesting two-pay auction mechanism was considered by Radosveta Ivanova-Stenzel and Doron Sonsino (Ivanova-Stenzel, et al., 2004). Based on the Israel experience of "The State Auction", authors consider the model of auction with several bids. Auction rules are the following: each bidder offers two bids. As for first-price sealed-bid auction, the bidder with the highest bet wins the object and if two or several bidders submit the same highest bid then the winner is chosen randomly with equal probabilities. The winner can pay his low-bid if this bid is higher than high-bids of all other bidders. Otherwise, the winner pays his high-bid. Authors found the equilibria for this auction, computed the expected revenue of auctioneer and compared it with the standard English auction.

Multi-stage auctions, as was mentioned before, usually applied in the case when several objects are sold. In this case, the auction is called sequential auction and there is quite a lot of literature in this case.

Sequential auction

Sequential auctions were studied by using different approaches. There are researchers where the effect of the different number of objects demanded by buyers is described. From this perspective, the multi-stage auction could be divided into the auction with multi-unit demand or singe-unit. Another issue, which is under study, is the effect of revaluation after disclosure of some information (for example value of bids) after some stages of the auction. As in the case of single round auction, there are works where the case of the independent value of bidders and the case of affiliated values are studied.

Firstly, equilibria of sequential auctions were found in the paper by Milgrom and Weber (Milgrom, et al., 1982). In their research, they show that prices at the next stages depend on prices at the previous stage (so-called martingale property). Also, they found that this property was violated in the case of interdependent values and, moreover, prices tended to drift upward. On the contrary, Mezzetti, Peke and Tsetlin (Mazzetti, et al., 2007) derived a symmetric equilibrium in case when part of objects was sold in the first stage and another part in the second stage. They considered auctions with revelation of the bet which won at the first-stage. Authors show that sequential auction possesses the effect of reducing the winning first-round price (so called "lowballing effect"). They prove that a total revenue of auctioneer is higher for single-round auction than for sequential auction in case of no announcement winning bid. But they notice that such type of auctions with the announcement of winning bid could have greater revenue than one-shot auction, under specific types of model.

A lot of papers discover the impact of different binding conditions on the expected revenue of auctioneer. For example, Caillaud and Mezzetti (Caillaud, et al., 2004) study the model of sequential, second-price auctions and the impact of different levels of the reserve price on "shifting" equilibria from the standard second-price auction. A reserve price is set up exogenously by seller before the beginning of auction. Authors characterize the equilibrium of the auction in case when seller has not any commitment from his side. Different types of bidder follow strategies which provides security of information. For example, low-valuation bidders don't take part in the early auction and high-valuation bidders tend to "overvalue" their true valuation by the bidding higher bets. The buyers, who receive a profit by buying at the reserve price, try to avoid from participation at the first auction with purpose to decrease the reserve price at the second-auction. As a result, authors summarize that there is no symmetric, monotone, pure-strategy equilibrium in which a positive measure of types is revealed at the end of the first auction. There are findings of the impact of bidder's constraint on auctioneer's revenue. For example, Saini (Saini, 2010) considers the model with n risk-neutral bidders, with independent private cost and constraints each of whom wishes to win both contracts. The auctioneer imposes a reserve price of R in the first auction. In every round, the bidders simultaneously submit sealed bids, and the contract is awarded to the lowest bidder. In case when there are at least two bidders who offered the lowest bid, then the auctioneer signs the contract to each bidder with equal probability. In the article, it is shown the existence of the Perfect Bayesian Nash Equilibrium for this auction and it is found for two-period auction.

Some researchers consider the special case of additional requirements for bid in case of procurement auction. Goswami (Goswami, 2013) based on the Indian experience in the procurement with quality competition, considers the model of the auction where bidders (suppliers) offer two "bids" – price and quality of the good. This auction is two-stage auction: in the first stage bidders with top quality are qualified to the second stage where the bidder with the lowest price will win and should deliver the good with the declared quality. The author shows that this procurement auction has no symmetric equilibrium within continuous monotonic pure strategies. He makes a conclusion that the auction with exogenously defined reserve quality is better than the auction mechanism where a minimal quality level is determined during the auction.

The first results obtained for multi-unit demand in sequential auctions are found by Katzman (Katzman, 1999). The author considers two very standard second-price sealed-bid auctions, where each individual bidder has diminishing marginal valuations. He concludes that when there is no incompleteness of information, prices at the first and next following rounds are stable or tend to decrease. Additionally, author notices that the winning allocation of objects can be inefficient in this case. On the contrary, if the information is incomplete and symmetric, bidder's behavior leds to efficient allocation and prices have tendency to growth. These findings are reconciled by using the argument based on ex-ante bidder asymmetry. Katzman found a symmetric equilibrium for this model of the auction.

The effect of the information asymmetric on auctioneer revenue in the dynamic auction was derived by Wang (Wang, 1993). He considers two-round auctions and the situation when a seller has some private information about the bid. On the contrary, bidders don't have such information. The seller sets up reserve price for the first round and in this way, he indicates the character of his information (if the price is high then it's a good product, otherwise it is not). Results show that for a seller it is more profitable if the seller sets up reserve price, which allows selling this product in the first round.

Multi-round auction with one object

In a few papers, the situation of bidder's strategies and equilibrium for multi-round auctions, where only one object is sold, is described. Moreover, these models are very specific. For example, Hirata (Hirata, 2014) considered a special auction mechanism. The author set up a new model of all-pay auction with two players and two stages. All bidders submit two bids – at the first and after revealing at the second stage. They offer their bets simultaneously. The winning bid is defined as the maximum sum of the bets at each stage. But each player should pay his bid. The author shows the existence of the unique equilibrium in non-degenerate mixed strategies in such type auction and provides it in the explicit form. In addition, he finds the auctioneer's revenue in the auction and compares the auction with another mechanism like Stackelberg auction and remarks that these results could be easily extended to the case with more players.

In some sense, a similar auction to the auction mode, which I plan to consider, is examined by Virag (Virag, 2007). There, the author applies the model of the auction to the labor market by modeling the case when a worker sells his labor force to employers, as two-round repeated auction. The article examines a first price sealed-bid auction with asymmetry in the private information among bidders. The winner at each stage of the auction could employ the worker (object) without revealing his bets made in the previous periods. The author uses similar ideas which are used in static games for finding the equilibrium in this type of repeated auction. Author concludes that in the offered model, bidders tend to bid a higher bet in a repeated auction than in a static auction because a higher probability to win increases the amount of information which is gathered during the bidding. The more aggressive behavior of the bidders also constitutes that auctioneer's expected revenue in the repeated auction is greater than in the one-shot auction.

Chapter 3

METHODOLOGY

As mentioned in the previous chapters, we will try to compare revenue of the auctioneer in the standard first-price sealed-bid auction with a model of the two-stage auction. First, we should give a formal definition of the sealed-bid auction. According to Maschler (Maschler, et al., 2013), the definition of the sealed-bid auction is the next:

Definition (Sealed-bid auction): A sealed-bid auction (with independent private values) is a vector $(T, (V_i, F_i)\ i \in N, p, C)$, where:

- $T = \{1, 2, \ldots, n\}$ is the finite set of purchasers.
- $V_i \subseteq R$ are possible private values of the object for buyer i, for each $i \in N$.
- For each buyer $i \in N$ there is a cumulative distribution function F_i over his set of private values V_i.
- $p: [0;\ \infty)^N \to \Delta(N)$ is a function which attains to bids $b\epsilon[0;\ \infty)^N$ with a distribution according to which the buyer who wins the auctioned object is identified.
- $C: N \times [0;\ \infty)^N \to R^N$ is a function determining the payment each buyer pays, for each vector of bids $b\epsilon[0;\ \infty)^N$, depending on which buyer $i_*\epsilon N$ is the winner.

In a sealed-bid auction, each player (bidder) follows the next rules:

- The value of object v_i of each buyer i, is chosen randomly from the set V_i, according to the cumulative distribution function F_i.

- Each bidder i knows his private value v_i, but he doesn't know the value of other bidders.

- Every bidder i chooses a bid $b_i \epsilon [0; \infty)$ with respect to his private value v_i.

- The winner, i_*, is chosen according to the distribution $p_i(b_1, b_2, \dots, b_n)$; the probability that buyer i wins the object is $p_i(b_1, b_2, \dots, b_n)$.

- Every buyer i pays the sum $C_i(i_*; b_1, b_2, \dots, b_n)$.

The case of the first-price sealed-bid is the symmetric auction with independent private values. This type of auction we will consider as the first stage for our model of the auction. Below, we provide a definition of such type of auctions:

<u>Definition (Symmetric auction with independent private values)</u> (Maschler, et al., 2013): First-price sealed-bid auction, which satisfies conditions below, is called symmetric auction with independent private values:

1. Only one object offered for sale in the auction, and its object couldn't be sold in the parts is indivisible.

2. The auctioneer ready to sell the object of the auction only at positive price.

3. There are n buyers.

4. Private values: All buyers have the same set of possible private values V. This set can be a closed bounded interval $[0, v]$. Every buyer knows his private value of the object. The random values $v_1, v_2, \dots, v_n$ of the private values of the buyers are independent and identically distributed. Denote by F the common cumulative distribution function of the random variables v_i , $i = 1, \dots, n$.

5. Continuity: For each i, the random variable v_i is continuous, and its density function, which we denote by f, is continuous and positive.

6. Risk neutrality: All the buyers are risk neutral. The main goal for each buyer is the maximization of his expected profits.

In our model of the auction, we will consider a two-period auction, which consists of the symmetric auction with independent private values in the first stage and specially sequenced auction in the second stage. For full description of our model we need to add next conditions:

<u>Model of the two-stage auction:</u>

1. There are 2 buyers.

2. Buyers have the same set of possible private values V. Each player knows his private value $v_{\mathrm{i}}, i = 1,2$ and random variables which modelled these values are identically and independently distributed.

3. The common cumulative distribution function of the random variables v_i is the uniformly distributed random variable on the segment $[0, 1]$.

4. In the first stage, each bidder i submits their bid $b_i^1 \epsilon [0, v_i]$ and for second stage $max\{b_1^1, b_2^1\}$ become a reserve price r.

5. There is a priority of bidding for the second stage: the bidder with the lowest bet will make his choice first, the bidder with the second highest price will bid second and the player with the highest price at the first stage will have the opportunity to pay the b_2^2 or reject it and in this case the bidder with b_2^2 – bet will become the winner.

6. Every bidder is risk-neutral and he tries to maximize his expected payoff in the whole auction. At the first round, each player chooses his strategy so as increase his probability to win in the second-round given his considerations and beliefs about another player.

7. If player receive offer to accept the bet which equals to his maximum valuation of the object he tends to reject this offer.

All conditions mentioned above mean that when the buyer $i's$ with his private value v_{i}, then if he wins the auctioned object at price p, his profit is $v_{\text{i}} - p$, whether he knows the private values of other buyers.

Note that for the second stage we have some asymmetry in information, due to the reason that the player with the highest bid knows less than other players since all participants of auction know only the reserve price, in other words "winner" of the first stage has the uniformly distributed valuation on $[r,\ 1]$.

But at the first stage we observe situation of symmetric auction, when each player doesn't have more information about another player or about object at all. For case of symmetric auction, as in case of first-price sealed-bid auction (Maschler, Solan and Zamir 2013), is very natural expect to receive symmetric equilibrium. At the same time, we should notice that in offered model two rounds are interconnected, and even expectation to receive more information in the first round could effect on the symmetry of this round. Below, we provide a definition of equilibrium in the symmetric auction.

Definition (Symmetric equilibrium) (Maschler, et al., 2013): In a symmetric auction with independent private values, an equilibrium $(\beta_1^*, \beta_2^*, \dots, \beta_n^*)$ is called a symmetric equilibrium $\beta_i^* = \beta_j^*$ for all $1 \leq i, j \leq n$; that is, all buyers implement the same strategy.

The proposed model of the auction is described by the concept of the dynamic game of incomplete information. A common approach to finding equilibrium in this type of games is finding of perfect Bayes equilibrium (PBE). This is an extending of Bayes equilibrium, which is equilibrium for the dynamic game with complete information. Gibbons (Gibbons, 1992) cited the following definition of the PBE:

Definition (PBE): A perfect Bayesian equilibrium consists of strategies and beliefs satisfying the next four requirements:

Requirement 1. The player needs to make movement should have beliefs at which node game already reached at this information set. For a nonsingleton information set, a belief is a probability distribution over the nodes in the information set; for a singleton information set, the player's belief concentrates all probability on this unique decision node.

Requirement 2. For each bidder, his strategy should be sequentially rational with given their beliefs. Consequently, the player chooses his strategy in an optimal way at each node, considering his beliefs and full set of actions of other players after this node.

Requirement 3. If information set is located on the equilibrium path then player's beliefs are satisfied to Bayes' rule and is a result of the players' optimal strategies.

Where information set on equilibrium path is defined by the next definition:

Definition (Information Set on Equilibrium Path): We will call information set, which could be reached with non-negative probability if players follow equilibrium strategies, as an information set is on the equilibrium path. In another case, information set is off the equilibrium path.

Requirement 4. If information set is located off the equilibrium path then player's beliefs are satisfied to Bayes' rule and is a result of the players' optimal strategies where it is possible.

Remark. The offered model of two-round auction quite well satisfies to the definition of signaling game. Really, if we consider player's bid $b_i^1, i = 1,2$ at the first-round as the signal for another player and player's bid b_i^2 at the second-round as the action in the game, and assume that each player i with probability $1 > p_i > 0$ expects that another j player has lower valuation of the object $v_i > v_j$ and with probability $1 - p_i$ vice versa, this game will be signaling game.

Since we will compare strategies of the bidders and auctioneer's expected revenue in proposed two-round auction with first-price sealed-bid auction, we need to provide similar results for first-price sealed-bid auction in case of two players. According to proof in (Maschler, et al., 2013), symmetric equilibrium for first-price sealed-bid auction with two players is strategy profile $(b_1, b_2) = (\frac{v_1}{2}, \frac{v_2}{2})$. Indeed, let's assume that second player follows strategy $b_2 = \frac{v_2}{2}$. The first player has the motivation to deviate from strategy $b_1 = \frac{v_1}{2}$ if this in some way increase his payoff. The next expression gives his expected profit:

$$\Pi\left(b_1, \frac{v_2}{2}, v_1\right) = P\left\{b_1 > \frac{v_2}{2}\right\}(v_1 - b_1) = P\{2b_1 > v_2\}(v_1 - b_1) = \\ = \min\{2b_1, 1\}(v_1 - b_1) \tag{1}$$

The profit function is quadratic over the interval $b_1 \in [0, \frac{1}{2}]$ with maximum at $b_1 = \frac{v_1}{2}$. On the interval $b_1 \in [\frac{1}{2}, 1]$, this function is linear with the negative slope, that's why again the profit function attains maximum at the point $b_1 = \frac{v_2}{2}$. The last means that this strategy is the best response for the first player. And whereas bidders are symmetric, the same reasons the second player shouldn't deviate from strategy $b_2 = \frac{v_2}{2}$. It proves, that strategy profile $(b_1, b_2) = (\frac{v_1}{2}, \frac{v_2}{2})$ is the symmetric equilibrium. This result will be our benchmark for the comparison player's behavior and equilibrium in our model.

Chapter 4

EXPLANATION OF BIDDER'S BEHAVIOR

Since we have an auction with two stages, for finding equilibrium in the model we need to use the backward induction method, in other words, we need to start from the end of the game. Let's consider the second stage of our auction. As it follows from the model of the auction we have 2 players with their valuations $v_1,\ v_2$ and reserve price $r = \max\{b_1^1, b_2^1\} = \ b_1^1$. But contrary to the previous stage there is the situation with asymmetric information, because in the first round the first player reveals that his valuation of the object is not less a r. Hence, the second bidder knows and the first bidder knows that the second bidder knows that his valuation $v_1 \in U[r,\ 1]$, in return the first player knows about the second and the third player only that their valuation $v_2\ \in U[0,\ 1]$.

We should make one important remark about the rational behavior of the bidders and their expectations:

<u>Remark:</u> If the first player submits bid $b_1$in the first round and $b_1^1 >\ b_2^1$, it means for the second player that valuation of the first player is higher than valuation of the second player or $v_1 > v_2$.

The last remark is applicable and for the first player, in other words, if $b_1^1 >\ b_2^1$ he assumes that $v_1 > v_2$. We can show that the linear strategy profile $[\frac{v_1}{2};\ \frac{v_2}{2}]$ is not unique optimal in the first round for this case. This means that for both players there is another strategy, which doesn't decline the payoff of each player.

Let's show it. By contradiction, let's assume that players follow $[\frac{v_1}{2}; \frac{v_2}{2}]$ strategy profile in the first round. As suggested by the perfect Bayes equilibrium approach, we should start from the second stage for proving that $[\frac{v_1}{2}; \frac{v_2}{2}]$ is optimal. In the second round there could be two cases, depending on the relation between v_1 and v_2:

1. A bid of the first player in the first round is higher than the valuation of the second player or $v_2 < \frac{v_1}{2}$;

2. A bid of the first player in the first round is less than the valuation of the second player or $v_2 > \frac{v_1}{2}$.

In the first case, the second player will submit nothing except the reserve price $r = \frac{v_1}{2}$ or even reject to submit anything. It befalls, because the price is too high for him and if he wins even with reserve price, his payoff will be negative. But since the first player is rational and he offers in the previous round the bid $b_1^1 = \frac{v_1}{2}$, which becomes the reserve price for the second stage it means that he is ready to take the object for $r = \frac{v_1}{2}$. Consequently, we have for the second round the equilibrium profile $["take"; r]$ for this case.

The second case is more complicated. From $v_2 > r$ and the second players assumption that $v_2 < v_1$ it follows that the second player is indifferent choosing the bid $b_2^2 \epsilon [r; v_2]$. He is afraid to offer $b_2^2 > v_2$ due to the incredible threat. Even since he expects that $v_2 < v_1$ he doesn't know the difference between v_2 and v_1, which means he will not try to offer the bid higher than v_2 for not overbidding the v_1. The latter means that in the second round all outcomes $["take"; b_2^2]$ where $b_2^2 \epsilon [r; v_2]$ are possible and optimal for players. Let's notice that payoffs for the second player equal to 0 and for the first player is varied in interval $[v_1 - v_2; \frac{v_1}{2}]$ in this case.

Now, let's return to the first round. We need to consider the strategy of the first player for given outcomes in the second round and given strategy of the second player $b_2^1 = \frac{v_2}{2}$ in the first round. The first bidder faces the next maximization problem:

$$\max_{b_1^1}(v_1 - b_2^2(b_1))P\{b_2^2(b_1^1) < \\ < v_1 | b_1^1 > b_2^1\} + (v_1 - b_1^2(b_2^1))P\{b_1^2(b_2^1) \geq \\ \geq v_2 | b_1^1 < b_2^1\} \quad (2)$$

under constraint $b_1^1 \leq b_2^2 \leq v_2$

The last formula means that the first player tries to maximize his payoffs in the auction by choosing the strategy in the first round under the condition that he is really "first player" (first term of the whole expression) and the condition that he lost the first round, but, nevertheless his valuation is the highest valuation (second term of the expression). The detailed description and solution to this problem will be made in the next part. Now, we should remember, that the payoff of the first bidder depends on the bid of the second player in the second round, which in its turn depends on the bid of the first player in the first stage. Consequently, we should return to two cases of the second round. In the case $v_2 > \frac{v_1}{2}$, the second player could play "aggressive" strategy ($b_2^2 = v_2$), "peaceful" strategy ($b_2^2 = r$) or their mix ($b_2^2 = tr + (1-t)v_2$) since he is indifferent between them. First, we try to consider two extremal cases ("aggressive" and "peaceful" strategy) for understanding how the first player should choose his bid in the first round.

Case of "aggressive" strategy

Let's assume that the second player plays the aggressive strategy in the second stage and $v_1 - v_2 < \frac{v_1}{2}$. Here the first player can't increase his payoff from $v_1 - v_2$, because if he chooses $b_1^1 > v_2$ he will decrease his benefit, but if he chooses $b_1^1 < \frac{v_2}{2}$ he will lose in the first round. The latter means that he is indifferent to choose between $\frac{v_2}{2}$ and v_2. As a result, $b_1^1 = \frac{v_1}{2}$ is not unique optimal.

In the case if $v_1 - v_2 > \frac{v_1}{2}$, the first player could increase his payoff from $v_1 - b_2^2(b_1^1) = v_1 - \mathrm{r} = v_1 - \frac{v_1}{2} = \frac{v_1}{2}$ by bidding lower bid b_1^1 up to v_2. But at the same time, the first player increases the risk to lose in the first round, which means that his expected payoff will decrease. The last means that for the case $v_2 < \frac{v_1}{2}$, it's not obvious if the strategy $b_1^1 = \frac{v_1}{2}$ is optimal for the first player or not.

Case of "peaceful" strategy

Since in this case the second player always chooses to bid the reserve price, the first player has only one constraint $b_1^1 \geq \frac{v_2}{2}$ for the purpose to remain a winner in the first round. That's why he has an infinitely great number of possible bids ($b_1^1 \geq \frac{v_2}{2}$ and $b_1^1 < \frac{v_1}{2}$), which brings him an higher payoff than strategy $b_1^1 = \frac{v_1}{2}$. The last one means that this strategy could be not optimal.

Whereas, the case of $v_2 < \frac{v_1}{2}$ is reduced to the case of "peaceful" strategy. The last one proves that for the first player the strategy $b_1^1 = \frac{v_1}{2}$ is not unique optimal.

Chapter 5

CASE OF "AGGRESSIVE" STRATEGY

Now, we try to answer the question, if it is symmetric equilibrium in the whole game and the first round particularly. Primarily, we should check, if the strategy $b_i^1 = \frac{v_i}{2}$ for the first round, could be the part of the equilibrium of the whole game.

Second round

Let's assume, that it is true and the player with the lower valuation v_2 follows this strategy. Let's check if the first player also follows this strategy. It's not difficult to show that for the player with lower valuation of the object, the strategy $b_2^2 = v_2$ is optimal in the second round in this case. The second player, if he believes that all players follow in the first round the symmetric strategy, it will maximize his expected payoff in the second round:

$$E[\Pi\left(z, v_2, b_2^1 = \tfrac{v_2}{2}\right)] = F_2\left(z \middle| b_2^1 = \tfrac{v_2}{2}\right) \times [v_2 - b_2^2(z, \tfrac{v_2}{2})], \tag{3}$$

Where F_2 is the distribution function of winning for the second player, and z is the value for which the bid (strategy) b_2^2 of the second player is optimal for him at the second stage.

For solving this problem, we apply the first order condition: differentiating $\Pi\left(z, v_2, b_2^1 = \frac{v_2}{2}\right)$ with respect to z:

$$f_2\left(z\middle|b_2^1=\frac{v_2}{2}\right)\times\left[v_2-b_2^2\left(z,\frac{v_2}{2}\right)\right]-\ F_2\left(z\middle|b_2^1=\frac{v_2}{2}\right)\times b_2^{2'}(z,\frac{v_2}{2})=0, \qquad (4)$$

Where f_2 is the density of the distribution function F_2. Reformulating this result in the differential equation form:

$$b_2^{2'}(z,\frac{v_2}{2})=\frac{f_2\left(z\middle|b_2^1=\frac{v_2}{2}\right)}{F_2\left(z\middle|b_2^1=\frac{v_2}{2}\right)}[v_2-b_2^2\left(z,\frac{v_2}{2}\right)] \qquad (5)$$

With natural, but not so relevant for this case boundary condition $b_2^2\left(0,\frac{v_2}{2}\right)=0$.

One of the solutions of the equation above is the strategy $b_2^2\left(z,\frac{v_2}{2}\right)=v_2$. Really, this is a stable outcome for the second round, because there could be 2 situations in the first round:

1. The second player lose: $b_2^1<b_1^1$;
2. The second player won: $b_2^1>b_1^1$.

In the first situation, the second player realizes that the first bidder has higher valuation, consequently, he can't do better than bidding $b_2^2=v_2$. Indeed, the winner of the first round knows that he is the winner, and that he has higher valuation than the loser. In the second stage, the first player will be waiting for the move of the second player. But whatever action is chosen by the second player, he knows that the first player has higher valuation v_1 and, consequently, the first player agreed to pay the bid of the second player, which obviously couldn't be higher than v_2 due to the incredible threat.

Another case, if the second player won in the first stage, it means that the first player learned valuation v_2 of the second player, since he knows that the second player played $b_2^1 = \frac{v_2}{2}$ in the first stage. As a result, he can bid exactly v_2 at the second stage. For clarity, we can assume that in this situation the second player may well to agree on v_2 price and he will give up fighting for the object. In this case, the second player again receives 0 payoff and the first bidder $v_1 - v_2$. No one can do better under these assumptions.

We still consider the case when $v_2 < b_1^1$ or in other words, when valuation of the bidder who evaluate the object lower than another one is lower than the bid of bidders who evaluates the object higher. But this is the special case ("peaceful strategy") with the other resulted strategy, which will be considered later.

Remark: Now, when we show that the player with lower valuation will play strategy $b_2^2 = v_2$ if he bid $\frac{v_2}{2}$ at the first stage, it should be noticed that the player with higher valuation will always play strategy "agree" in this case, because his payoff $v_1 - v_2$ will be positive in this case.

First round

In the first-period, bidders are faced with slightly more complex decision-making problem. Now we assume, whatever may happen at the first stage, players follow in the second round the strategy profile [b_1^2= "agree"; $b_2^2 = v_2$]. And as before, one player follows the strategy $b_2^1 = \frac{v_2}{2}$. Now, if another player plays the same strategy it means that we found symmetric equilibrium.

It's necessary to notice that from our consideration of the second round the next statement naturally implies:

Statement (1): If one from these two players follows the strategy $b_i^1 = \frac{v_i}{2}$ in the first round and it is the player with lower valuation, then in the auction the player with higher valuation will win.

Under these circumstances, the first player will face the next maximization problem:

$$\max_{b_1^1} \Pi(b_1) = \mathrm{E}(v_1 - b_2^2 | v_1 > v_2)\, P\{b_1^1(v_1) > b_2^1(v_2)\} + \\ + \mathrm{E}(v_1 - b_1^2 | v_1 > v_2) P\{b_1^1(v_1) < b_2^1(v_2)\} + \\ + \mathrm{E}(v_1 - b_2^2 | v_1 < v_2) P\{b_1^1(v_1) > b_2^1(v_2)\} + \\ + \mathrm{E}(v_1 - b_2^2 | v_1 < v_2) P\{b_1^1(v_1) < b_2^1(v_2)\} \tag{6}$$

The expression above describes all possible outcomes of the auction for the first bidder. The first term describes the case when the player with higher valuation chooses his bid b_1^1 in the first round so that it is higher than the bid of the second player. The second term describes the opposite situation. The third and fourth terms describe the cases when the first player has lower valuation ($v_1 < v_2$) of the object and the first player wins and losses in the first round respectively. The profit-maximization expression could be simplified to the next:

$$\max_{b_1^1} \Pi(b_1) = \mathrm{E}(v_1 - b_2^2 | v_1 > v_2)\, P\left\{b_1^1(v_1) > \frac{v_2}{2}\right\} + \\ + \mathrm{E}(v_1 - b_2^2 | v_1 > v_2) P\left\{b_1^1(v_1) < \frac{v_2}{2}\right\} \tag{7}$$

Here, the third and fourth terms equal to 0, because we have a conditional expectation $\mathrm{E}(v_1 - b_2^2 | v_1 < v_2)$ in both terms, which under the statement (1) equals to 0. From the statement (1) we conclude that the player with higher valuation will win in the auction. But for $\mathrm{E}(v_1 - b_2^2 | v_1 < v_2)$ we observe that

$v_1 < v_2$, or in other words the first player has lower valuation than the second one, consequently, the first player will lose the auction and his payoff equals to 0.

In addition, since the first player assumes that the second player follows the strategy $b_2^1(v_2) = \frac{v_2}{2}$ we substituted in (1) $b_2^1(v_2)$ on $\frac{v_2}{2}$. Moreover, according to our consideration for the second round, the second player will bid in the second round the bid which exactly equals to his valuation. Hence, for the first the term in (7): $b_2^2 = v_2$. But even if the first player lost the first round and the observed valuation of the second player, and realized that his valuation is higher, the first player can bid exactly v_2. That's why, for the second term $b_1^2 = v_2$. As a result, (7) is simplified to:

$$\max_{b_1^1} \Pi(b_1) = \mathrm{E}(v_1 - v_2 | v_1 > v_2)\, P\left\{b_1^1(v_1) > \frac{v_2}{2}\right\} + \\ + \mathrm{E}(v_1 - v_2 | v_1 > v_2) P\left\{b_1^1(v_1) < \frac{v_2}{2}\right\} \tag{8}$$

But, since $P\left\{b_1^1(v_1) < \frac{v_2}{2}\right\} = 1 - P\left\{b_1^1(v_1) > \frac{v_2}{2}\right\}$, we have:

$$\max_{b_1^1} \Pi(b_1) = \mathrm{E}(v_1 - v_2 | v_1 > v_2)\left(P\left\{b_1^1(v_1) > \frac{v_2}{2}\right\} + \\ + P\left\{b_1^1(v_1) < \frac{v_2}{2}\right\}\right) = \mathrm{E}(v_1 - v_2 | v_1 > v_2) \tag{9}$$

The last expression shows us that whatever strategy a player with the highest valuation will choose in the first round if the second player follows $b_2^1 = \frac{v_2}{2}$ it does not change the expected payment of the player with the highest valuation.

Let's compute the expected payoff of the first player:

$$E(v_1 - v_2 | v_1 > v_2) =$$

$$= \frac{1}{P\{v_1 > v_2\}} \cdot$$

$$\cdot \left(\iint_D (v_1 - v_2) f_{v_1 - v_2}(v_1, v_2) dv_1 dv_2 \right) = \qquad (10)$$

$$= \frac{1}{P\{v_1 > v_2\}} \cdot \left(\iint_D (v_1 - v_2) f_{v_1 - v_2}(v_1, v_2) dv_1 dv_2 \right)$$

For computing the last expression, we need to notice that $f_{v_1 - v_2}(v_1, v_2)$ – density function of the difference of independent and uniformly distributed random variables, equals to

$$f_{v_1 - v_2}(x) = \begin{cases} x, if \ x \in [-1; 0] \\ 1 - x, if \ [0; 1] \\ 0, if \ x \notin [-1; 1] \end{cases} \qquad (11)$$

As a result:

$$\iint_D (v_1 - v_2) f_{v_1 - v_2}(v_1, v_2) dv_1 dv_2 =$$

$$= \left(\int_{-1}^{0} x^2 dx + \int_{0}^{1} x(1 - x) dx \right) = 1/2 \qquad (12)$$

At the same time

$$P\{v_1 > v_2\} = P\{v_1 - v_2 > 0\} = \int_0^1 (1 - x) dx = 1/2 \qquad (13)$$

Finally:

$$\mathrm{E}(v_1 - v_2 | v_1 > v_2) = 1/4 \tag{14}$$

It should be noticed from the point of view of the bidder with lower valuation, under the same consideration, namely that another player will play the strategy $b_1^1 = \frac{v_1}{2}$ in the first round, and after observing that he has higher valuation he will play the strategy $b_1^2 = v_2$ (by offering this price or simply agreeing on it) in the second round, he has no any incentives to deviate from the strategy $b_2^1 = \frac{v_2}{2}$. Indeed, if the bidder with lower valuation in some way overbid the player with higher valuation in the first period – another player will play his valuation $v_1 > v_2$ in the second period and will win the auction. From the other side, if the player with lower valuation loses the first round and will play v_2 in the second round the player with higher valuation will agree on this price and win the object.

As a result, for this case, we can conclude that under the assumption that in the second round the strategy profile [b_1^2= "agree"; $b_2^2 = v_2$] takes place, it's not important what strategies are chosen by players in the first round.

Chapter 6

CASE OF "PEACEFUL" STRATEGY

Now, let's consider the more complicated case when the player with expected lower valuation will use other approach of solving his profit-maximization problem. We already explained that when the player with lower valuation observed that the first player's bid in the first round is higher than his valuation, he will not bid in the second round anything except the reserve price.

Given this information, the player with higher valuation will face the similar problem in the first round, as in the case when the player with lower valuation plays v_2 in the second round:

$$\begin{aligned}\max_{b_1^1} \Pi(b_1) = \mathrm{E}(v_1 - b_2^2 | v_1 > v_2)\, P\{b_1^1(v_1) > b_2^1(v_2)\} + \\ + \mathrm{E}(v_1 - b_1^2 | v_1 > v_2) P\{b_1^1(v_1) < b_2^1(v_2)\} + \\ + \mathrm{E}(v_1 - b_2^2 | v_1 < v_2) P\{b_1^1(v_1) > b_2^1(v_2)\} + \\ + \mathrm{E}(v_1 - b_2^2 | v_1 < v_2) P\{b_1^1(v_1) < b_2^1(v_2)\}\end{aligned} \tag{15}$$

But here, we have another outcome. With similar arguments, as in the previous case, the third and fourth terms of the last expression will wane. In the first term $b_2^2 = r = b_1^1(v_1)$ because the second player observed that overbid b_1^1 is too expensive for him. Since the second term describes the case when the first player lost, but observed that his valuation is higher than the second player valuation, the first player will bid $b_1^2 = v_2$ for overbidding the second player. Hence, we have:

$$\max_{b_1^1} \Pi(b_1) = \mathrm{E}(v_1 - b_1^1(v_1)|v_1 > v_2)\, P\left\{b_1^1(v_1) > \frac{v_2}{2}\right\} + \\ + \ \mathrm{E}(v_1 - v_2|v_1 > v_2) P\left\{b_1^1(v_1) < \frac{v_2}{2}\right\} \tag{16}$$

The last maximization problem is solved by taking the derivative with respect to b_1^1. But first of all, we ought to notice that it's natural to assume, that $b_1^1(v_1)$ it increases the function and guarantes that for this function there is the inverse function $(b_1^1(v_1))^{-1}$ according to the theorem about the continuous and increasing function. Since $P\left\{b_1^1(v_1) < \frac{v_2}{2}\right\} = 1 - \left\{b_1^1(v_1) > \frac{v_2}{2}\right\}$, and fact that $P\left\{b_1^1(v_1) > \frac{v_2}{2}\right\} = P\{v_2 < 2b_1^1(v_1)\} = G(2b_1^1(v_1))$, where $G(\cdot)$ is the distribution function of v_2. For simplicity, we will denote $b_1^1(v_1) = b_1^1$. We can rewrite the profit-maximization expression:

$$\max_{b_1^1} \Pi(b_1) = \mathrm{E}(v_1 - b_1^1|v_1 > v_2)\left(1 - G(2b_1^1)\right) + \\ + \ \mathrm{E}(v_1 - v_2|v_1 > v_2) G(2b_1^1) \tag{17}$$

Simplifying the last expression:

$$\max_{b_1^1} \Pi(b_1) = \mathrm{E}(v_1 - b_1^1|v_1 > v_2)\left(1 - G(2b_1^1)\right) + \\ + \ \mathrm{E}(v_1 - v_2|v_1 > v_2) G(2b_1^1) = \\ = \mathrm{E}(v_1 - b_1^1|v_1 > v_2) + \ \mathrm{E}(b_1^1 - v_2|v_1 > v_2) G(2b_1^1) \tag{18}$$

If applying the first order condition with respect to b_1^1, we will receive:

$$\left(-\,\mathrm{E}\left(b_1^{1\prime}\middle|v_1 > v_2\right)\right) + \\ + 2(\mathrm{E}(b_1^1|v_1 > v_2) - \mathrm{E}(v_2|v_1 > v_2)\,)g(2b_1^1) + \\ + \mathrm{E}\left(b_1^{1\prime}\middle|v_1 > v_2\right)G(2b_1^1) = 0 \tag{19}$$

Making simple arithmetic transformation, we will receive:

$$\mathrm{E}\left(b_1^{1\prime}\middle|v_1 > v_2\right)(\,G(2b_1^1) - 1) + 2\mathrm{E}(b_1^1 - v_2|v_1 > v_2)g(2b_1^1) = 0 \tag{20}$$

Any conditional expectation from b_1^1 and its derivative equal to itself, because b_1^1 is not a random variable, that's why, the last expression could be simplified to the next expression:

$$b_1^{1\prime}(\,G(2b_1^1) - 1) + 2b_1^1 g(2b_1^1) - 2\mathrm{E}(v_2|v_1 > v_2)g(2b_1^1) = 0 \tag{21}$$

Since in our case we choose $G(\cdot)$ as uniformly distributed function on the segment [0; 1], we should to consider different cases depending on the value of $2b_1^1$:

$$G(2b_1^1) = \begin{cases} 2b_1^1, if\ 2b_1^1\ \in [0;1] \\ 1, \quad\quad if\ 2b_1^1 > 1 \end{cases} \tag{22}$$

And consequently:

$$g(2b_1^1) = \begin{cases} 1, if\ 2b_1^1\ \in [0;1] \\ 0, \quad\quad if\ 2b_1^1 > 1 \end{cases} \tag{23}$$

But the case when $2b_1^1 > 1$, we receive the degenerate case (0=0), because $1 - G(2b_1^1) = 1 - 1 = 0$ and $g(2b_1^1) = 0$. In the case, when $2b_1^1\ \in [0;1]$ we will receive:

$$b_1^{1'}(2b_1^1 - 1) + 2b_1^1 - 2\mathrm{E}(v_2|v_1 > v_2) = 0 \tag{24}$$

But $\mathrm{E}(v_2|v_1 > v_2) = \frac{v_1}{2}$, because v_2 is the uniformly distributed random variable on the segment $[0; v_1]$ (due to condition $v_1 > v_2$) and according to the general formula for the expected value of the uniformly distributed random variable $\mathrm{E}(X) = \frac{a+b}{2}$, if $X \in [a; b]$. That's why, the last expression will reduce to:

$$b_1^{1'}(2b_1^1 - 1) + 2b_1^1 - v_1 = 0, \tag{25}$$

or

$$b_1^{1'} = \frac{v_1 - 2b_1^1}{2b_1^1 - 1} \tag{26}$$

The last equation is a Cauchy problem with the following initial condition $b_1^1(0) = 0$, which means that the bidder with the zero-valuation is ready to bid only 0.

It is obvious that the linear strategy $b_1^1(v_1) = \frac{v_1}{2}$ is not the solution to the last Cauchy problem, which means that the auction has no symmetric equilibrium in this case.

We can't find explicit solution of this equation, because it is a nonlinear and nonhomogeneous equation which can't be simplified. But we can find the approximated solution, which satisfies initial conditions. We will look for solution in the next form:

$$b_1^1(v_1) = b_1^1(0) + b_1^{1'}(0)v_1 + \frac{b_1^{1''}(0)}{2!}{v_1}^2 + \frac{b_1^{1'''}(0)}{3!}{v_1}^3 + o({v_1}^3) \tag{27}$$

The expression above is Taylor's series of the $b_1^1(v_1)$ function at the point $v_1 = 0$. The existence solution in such form is guaranteed by the assumption that $b_1^1(v_1)$ is analytical function at the $v_1 \in [0,1]$ (Titchmarsh 1939). Here $o({v_1}^3)$ is the all other elements in Taylor's sum which are higher by order compare with ${v_1}^3$. We can provide more accurate solution, but since we consider case when $v_1 \in [0,1]$, even approximation up to third order is well enough for making conclusions about $b_1^1(v_1)$.

According to initial condition $b_1^1(0) = 0$, we receive that the first term in Taylor's series equals to 0. Consequently, the second term equals to:

$$b_1^{1\prime}(0) = \frac{0 - 2b_1^1(0)}{2b_1^1(0) - 1} = \frac{0}{-1} = 0 \tag{28}$$

Calculations for third and fourth term in (27) are provided in the appendix by formulas (38) and (40). As the result, we obtain the next approximation for

$$b_1^1(v_1) = |-\frac{{v_1}^2}{2}| + o({v_1}^3) \tag{29}$$

This means that player's behavior deviates from the strategy profile of $[\frac{v_1}{2}; \frac{v_2}{2}]$, which is an equilibrium strategy profile for the first-price sealed-bid auction. Consequently, it implies that the auctioneer's expected revenue for the case of the two-round auction will be different. But for the explicit finding of the revenue, we should consider all the "mixed" cases, when the player with lower valuation doesn't play v_2 or reserve the price in the second round. In return, we can compare the auctioneer's expected revenue with revenue in the first-price sealed bid auction.

Chapter 7

COMPARE AUCTIONEER'S REVENUE IN TWO-ROUND AUCTION AND FIRST-PRICE SEALED-BID AUCTION

In this part, let us compare the results obtained for the two-round auction with the results for the first-price sealed-bid auction in terms of the auctioneer. According to (Maschler, Solan and Zamir 2013), in the case of the first-price sealed-bid auction with two players with uniformly distributed on segment [0,1] valuations, the auctioneer's expected revenue equals to 1/3.

We will start to compute the auctioneer's expected revenue from the case when the second player follows aggressive strategy in the second round. As is found in the previous chapter, if the player with lower valuation follows the strategy $b_2^2(v_2) = v_2$ only two possible outcomes can be in this case:

- player with lower valuation bids the bet equals to v_2 and the player with higher valuation agreed to pay this amount (the case when the player with higher valuation won in the first round)

- player with higher valuation bids the bet equals to v_2 and player with lower valuation rejects this bet, as a result the player with higher valuation wins the object by the price v_2 (the case when the player with higher valuation lost in the first round)

But in both cases, the highest payoff which auctioneer receives is equal to v_2. Consequently, the expected revenue equals:

$$E[\min\{V_1, V_2\}] \tag{30}$$

Where $V_i, i = 1,2$ distribution functions, which describe bidder's valuation.

We will compute (30), using the inverse expression - $\max\{V_1, V_2\}$. Note that:

$$\max\{V_1, V_2\} + \min\{V_1, V_2\} = V_1 + V_2 \tag{31}$$

Let's denote $X = \max\{V_1, V_2\}$. While we assume that distribution functions V_1, V_2 are independent and uniformly distributed on [0,1]:

$$\begin{aligned} F_X(x) = P(X < x) = P(\max\{V_1, V_2\} < x) = \\ = P(V_1 < x) \times P(V_2 < x) = x^2 \end{aligned} \tag{32}$$

The expected value of the last expression equals:

$$E[X] = \int_0^1 x dF_X(x) = \int_0^1 2\,x^2 dx = \frac{2}{3} \tag{33}$$

Since $E[V_1] = E[V_2] = \frac{1}{2}$, because they are uniformly distributed on [0,1], we receive:

$$E[\max\{V_1, V_2\}] + E[\min\{V_1, V_2\}] = E[V_1] + E[V_2] = \frac{1}{2} + \frac{1}{2} = 1 \tag{34}$$

And since we received that $E[\max\{V_1, V_2\}] = E[X] = \frac{2}{3}$, then $E[\min\{V_1, V_2\}] = \frac{1}{3}$. The last one means that in the "aggressive" case, the auctioneer's expected revenue equals to auctioneer's expected revenue in the first-price sealed-bid auction.

Remark. Since we received that in the case of the aggressive strategy winning the bet equals to valuation of the lower player, this result is the same as in the case of the second-price sealed-bid auction.

In the case when the player with lower valuation follows the peaceful strategy, or in other words he bids the bet equals to the reserve price and another player accepts this bid, we didn't get the explicit expression for the winning bet. But according to our estimation from the first round

$$r = b_1^1(v_1) = | - \frac{{v_1}^2}{2} | + o({v_1}^3) \tag{35}$$

Consequently, the winning bet equals to

$$\max\left\{ | - \frac{{v_1}^2}{2} | + o({v_1}^3), | - \frac{{v_2}^2}{2} | + o({v_2}^3) \right\} \tag{36}$$

This means that the winning bet in this case is smaller than in the first-price sealed-bid auction, where the winning bet equals to $\max\left\{\frac{V_1}{2}, \frac{V_2}{2}\right\}$ ($v_i \in [0,1]$ hence ${v_2}^2 \ll v_2$).

Since two extremal cases are considered, when the winning bet equals to the reserve price for the second round or to valuation of the player with lower valuation of the object and for both cases the auctioneer's expected revenue doesn't exceed the auctioneer's expected revenue in the first-price sealed-bid auction, we can conclude that the same result will be true for all mixed cases.

Chapter 8

CONCLUSIONS

Auctions are commonly used in various areas of the economy where there is no market or different types of market failures could be present. Depending on what goals are set up before the auctioneer, two main characteristics describe auctions: the expected revenue of an auctioneer and efficiency. There is more common practice for private auction that tries to maximize the expected revenue of an auctioneer. But it directly depends on strategies of players, which they follow. The last, in its turn, depends on rules and mechanisms of the proposed auction.

The goal of this research is to check whether the same strategy profile will be equilibrium for the multi-round auctions. In the paper, the model of the two-stage auction with two bidders was considered where only one object is sold. Each bidder is rational and risk-averse with a uniformly distributed valuation of the object. Since in the case of the multi-round auction we face a dynamic game of incomplete information, for finding the equilibrium of the auction we applied the mechanism of the perfect Bayes equilibrium. It is natural that we searched for player's strategies among linear strategies.

As was expected, the behavior of players in the first round is determined by the desired results in the second round. In this work, two types of bidders were considered: "aggressive" and "peaceful". According to the arguments provided in the paper, we can conclude that in the case of aggressive player's behavior, there is an infinitely great number of equilibrium strategies of the player in the first round, including the profile which is the equilibrium of the first-price sealed-bid auction.

For the case of peaceful player's behavior, the result is quite opposite. Despite the fact that we could not find the explicit form of equilibrium for this case, we clearly showed that the profile consisting of the halves the private players' valuation could not be the equilibrium for the first round.

There is another aspect concerning the expected profit auctioneers. Depending upon bidder's types it could equal to the auctioneer's expected revenue in the first-price sealed-bid auction (when a player with lower valuation relates to "aggressive" type), and it could be less if the player with lower valuation relates to another type. Therefore, we conclude that the first-price sealed-bid auction should bring higher payoff for the auctioneer than the offered two-stage model. This result is confirmed by the experiment of professor Spagnolo (Stepaniuk, 2017), which shows that the first-price sealed-bid auction brings higher expected revenue comparing to the two-round auction.

"Prozorro" makes use the similar, but more complicated auction mechanism than the auction considered in the paper. We examined one of the characteristics of the auction which is important for procurement auction – the expected revenue. The efficiency of the auction was not the topic of this research, but depending on the auctioneer's goals, for choosing the appropriate mechanism, this is quite important too. Relating to auctioneer's revenue, we received that even in the pretty simple case of two players with the uniformly distributed valuations, first-price sealed-bid auction is the more profitable mechanism for the auctioneer. Consequently, we can expect that for the more common situation, where the number of players is more than two, their valuations of the object are more sophisticated, the mechanism of the first-price sealed-bid auction is better for increasing of the auctioneer's profit. That's why, if the main task for "Prozorro" auction is to save more money for the state, it would be better to change the existed mechanism on another type of auction, for example on the mentioned first-price sealed-bid auction or second-price sealed-bid auction (because according to revenue equivalence principle these auctions are equivalent in the sense of auctioneer's revenue).

WORKS CITED

Birulin, Oleksii. 2003. "Inefficient Ex-Post Equilibria in Efficient Auctions." *Economic Theory*, October: 675-683.

Burguet, Roberto, and Jozsef Sakovics. 1999. "Imperfect Competition in Auction Designs." *International Economic Review* (40): 231-247.

Caillaud, Bernard, and Claudio Mazzetti. 2004. "Equilibrium Reserve Prices in Sequential Ascending Auctions." *Journal of Economic Theory* (117): 78-95.

Gibbons, Robert. 1992. *Game Theory for Applied Economists.* Princeton: Princeton University Press.

Goswami, Mridu Prabal. 2013. "Quality cut-offs in procurement auctions." *Economics Letters*, 148-151.

Hirata, Daisuke. 2014. "A Model of a Two-Stage All-Pay Auction." *Mathematical Social Sciences* (68): 5–13.

Ivanova-Stenzel, Radosveta, and Doron Sonsino. 2004. "Comparative Study of One-bid versus Two-bid Auctions." *Journal of Economic Behavior & Organization* (54): 561–583.

Katzman, Brett. 1999. "A Two Stage Sequential Auction with Multi-Unit Demands." *Journal of Economic Theory* (86): 77-99.

Maschler, Michael, Eilon Solan, and Shmuel Zamir. 2013. *Game Theory.* Cambridge: Cambridge University Press.

Mazzetti, Claudio, Aleksandar Sasa Pekec, and Ilia Tsetlin. 2007. "Sequential vs single-round uniform-price auctions." *Games and Economic Behavior*, 591-609.

Milgrom, Paul, and Robert Weber. 1982. "A Theory of Auctions and Competitive Bidding." *Econometrica* (50): 1089-1122.

Saini, Viplav. 2010. "Reserve Prices in A Dynamic Auction When Bidders Are Capacity-Constrained." *Economics Letters* (108): 303–306.

Stepaniuk, Oleksa. 2017. *What Is the Value of Prozorro Auction? Results of the Experiment/Чого вартий аукціон ProZorro? Результати експерименту.* Accessed March 2017 http://cep.kse.org.ua/article/choho-vartyy-auktsion-prozorro-Rezultaty-eksperymentu/index.html.

Titchmarsh, Edward. 1939. *The Theory of Functions.* 2-th. Oxford: Oxford University Press.

Virag, Gabor. 2007. "Repeated Common Value Auctions with Asymmetric Bidders." *Games and Economic Behavior* (61): 156–177.

Wang, Ruqu. 1993. "Strategic Behavior in Dynamic Auctions." *Economic Theory* (3): 501-5016.

APPENDIX

Calculation for the second term in Taylor's series for first player bid at the first stage:

$$b_1^{1''}(v_1) = \frac{\left(1 - 2b_1^{1'}\right)(2b_1^1 - 1) - 2b_1^1(v_1 - 2b_1^1)}{(2b_1^1 - 1)^2} \tag{37}$$

And $b_1^{1''}(v_1)$ at $v_1 = 0$ considering that $b_1^{1'}(0) = 0$ and $b_1^1(0) = 0$

$$\begin{aligned} b_1^{1''}(0) &= \frac{\left(1 - 2b_1^{1'}(0)\right)(2b_1^1(0) - 1) - 2b_1^1(0)(0 - 2b_1^1(0))}{(2b_1^1(0) - 1)^2} = \\ &= \frac{(1 - 0)(0 - 1) - 0(0 - 0)}{(0 - 1)^2} = -1 \end{aligned} \tag{38}$$

Calculation for the third term in Taylor's series for first player bid at the first stage:

$$\begin{aligned} b_1^{1'''}(v_1) = &\frac{-2b_1^{1''}(2b_1^1 - 1) - 4b_1^{1''}b_1^{1'} - 2b_1^{1''}(v_1 - 2b_1^1) - 2b_1^{1'}(1 - 2b_1^{1'})}{\left(2b_1^1 - 1\right)^4} - \\ &\frac{2\left(b_1^1 - 1\right)\left(\left(1 - 2b_1^{1'}\right)\left(2b_1^1 - 1\right) - 2b_1^{1'}\left(v_1 - 2b_1^1\right)\right)}{\left(2b_1^1 - 1\right)^4} \end{aligned} \tag{39}$$

And $b_1^{1'''}(v_1)$ at $v_1 = 0$ considering that $b_1^{1''}(0) = -1$, $b_1^{1'}(0) = 0$ and $b_1^1(0) = 0$

$$
\begin{aligned}
& b_1^{1'''}(0) = \\
& = \frac{-2b_1^{1''}(0)(2b_1^1(0) - 1) - 4b_1^{1''}(0)b_1^{1'}(0)}{(2b_1^1(0) - 1)^4} + \\
& + \frac{2b_1^{1''}(0)(2b_1^1(0) - 0) - 2b_1^{1'}(0)(1 - 2b_1^{1'}(0))}{(2b_1^1(0) - 1)^4} - \\
& - \frac{2(b_1^1(0) - 1)\left(1 - 2b_1^{1'}(0)\right)(2b_1^1(0) - 1)}{(2b_1^1(0) - 1)^4} + \\
& + \frac{4b_1^{1'}(0)(b_1^1(0) - 1)\left(0 - 2b_1^1(0)\right)}{(2b_1^1(0) - 1)^4} = \\
& = \frac{-2 \cdot (-1) \cdot (2 \cdot 0 - 1) - 4 \cdot (-1) \cdot 0 + 2 \cdot (-1) \cdot (2 \cdot 0 - 0)}{(2 \cdot 0 - 1)^4} - \\
& - \frac{2 \cdot 0 \cdot (1 - 2 \cdot 0)}{(2 \cdot 0 - 1)^4} - \\
& - \frac{2 \cdot (0 - 1) \cdot \left((1 - 2 \cdot 0)(2 \cdot 0 - 1) - 2 \cdot 0 \cdot (0 - 2 \cdot 0)\right)}{(2 \cdot 0 - 1)^4} = 0
\end{aligned}
\tag{40}
$$