ALTERNATIVE PORTFOLIO
DIVERSIFICATION APPROACHES:
AN EMPIRICAL COMPARISON

by

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Abstract

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In this study, we consider the multi-period portfolio optimization problem applying two methods: mean-variance (MV) and hierarchical risk parity (HRP). We intend to perform the comparative analysis of two approaches using 30 stocks of the DJIA over the years from 1990 to 2017. We take into account proportional transaction fees and assume that investor rebalances his portfolio each week with 5% threshold.

A studentized stationary circular bootstrap approach is applied to show that the difference between a particular indicator of MV and HRP is statistically significant.

The comparative analysis is conducted in three steps. First, we compare two methods over the entire period. Secondly, we repeat the previous procedure for four sub-periods. Lastly, we make the sensitivity analysis of portfolio performance depending on brokerage fees and thresholds. This allows us to make a deep comparative analysis.

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GLOSSARY

CAGR – compound annual growth rate.

CI – confidence interval.

DJIA – Dow Jones Industrial Average index. The market index that covers companies only with large capitalization and high liquidity.

EW – equally-weighted approach.

EWMA – exponentially weighted moving average method.

GARCH – generalized autoregressive conditional heteroskedasticity process.

HAC – heteroskedasticity and autocorrelation robust kernel estimation.

HRP – hierarchical risk parity method.

IVP – inverse-variance parity approach.

MV – minimum-variance method.

NASDAQ – National Association of Securities Dealers Automated Quotations, an American stock exchange.

NYSE – New York Stock Exchange.

QHRP – quantum-inspired hierarchical risk parity approach.

USA – United States of America.

VAR – vector autoregression model.

Chapter 1

INTRODUCTION

Optimal allocation of assets within a portfolio is a crucial problem for asset managers. In order to solve this problem, many portfolio diversification approaches have been developed. For instance, Markowitz (1952), Merton (1969), Magill and Constantinides (1976), Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991), Shreve and Soner (1994), Leland (2000), Donohue (2003), Pliska and Suzuki (2004), Muhle-Karbe (2016), and many other scientists investigated the portfolio optimization problem theoretically. In addition, there is a huge number of people who studied this problem empirically, including Baule (2008), Kritzman (2009), Gârleanu (2009), Lynch (2010), Ormos (2011), Zdorovenin (2012), Malamud (2014), Marasovic (2015), Lopez de Prado (2016), Alipour (2016), Ha (2017), Sağlam (2018) and others.

Overall, we can highlight the following well-known portfolio diversification approaches: mean-variance (MV), inverse variance parity (IVP), equally-weighted (EW), hierarchical risk parity (HRP) and others. Clarke (2006), Ormos (2011), Lopez de Prado (2016), Alipour (2016), Hautsch (2017) and other researchers conducted the comparative analysis of different portfolio diversification methods. Some researchers investigated the portfolio optimization problem without taking into account transaction costs, and some – with taking into account transaction costs. We are also aimed at conducting an empirical investigation of the portfolio optimization problem with transaction costs.

In this thesis we follow Pogue (1970), Chow (2006), Kritzman (2009), Gârleanu, Lopez de Prado (2016), Bielstein (2017) and choose the mean-variance approach as the benchmark portfolio diversification method for the comparative analysis. We compare this approach with the hierarchical risk parity approach recently suggested by Lopez de Prado (2016) and Alipour (2016). In particular, Lopez de Prado (2016) simulates the data to construct the portfolio using the MV, IVP, HRP approaches. However, Lopez de Prado (2016) and Alipour (2016) do not take into account transaction costs when they develop the portfolio so they provide sub-optimal portfolio.

Our purpose is to make our own contribution to the empirical investigation of the portfolio optimization problem making the comparative analysis of the MV and HRP approaches, similarly to Lopez de Prado (2016) and Alipour (2016). This thesis extends the results obtained by Lopez de Prado (2016) and Alipour (2016) by taking into account transaction costs in a similar fashion to Mitchell (2002). In particular, we check how MV and HRP portfolios perform given different brokerage fees, which is an important issue in portfolio management.

As a rule, investors aim at minimizing portfolio risk given a desired level of return. As a result, investors can find the set of optimal risky portfolios and draw the so-called efficient frontier of risky assets as Markowitz (1952) suggests. He developed a well-known method called the Mean-Variance (MV) approach, which is based only on two inputs: risk and returns of available assets. In addition, the investor has an opportunity to invest in the risk-free assets. Therefore, it is necessary to decide how much to invest in the risk-free and risky portfolios given his relation to risk. William Sharpe was the first to make a great contribution in solving this issue in 1966 by introducing a new risk-to-return measure called the Sharpe ratio. It describes how much risk premium investor gets for the additional volatility that he bears for holding the risky portfolio. Obviously, investors would like to get a compensation for the risk as much as possible. Thus, to construct a well-diversified portfolio, an asset manager should choose the risky portfolio on the so-called efficient frontier with the largest Sharpe ratio given his relation to the risk.

However, this classical approach has many drawbacks such as concentration, instability and underperformance out-of-sample (Lopez de Prado, 2016). The main reason why these problems exist is the form of the covariance matrix that, usually, has a high condition number that represents the ratio of maximal and minimal eigenvalues of the covariance matrix. In order to solve this problem, it is necessary to transform the covariance matrix to the diagonal form that provides us with the lowest condition number. Lopez de Prado (2016) was the first to implement the diagonalization procedure in his own portfolio diversification approach called the Hierarchical Risk Parity (HRP) approach. According to this method, the investor allocates his funds across all assets within each cluster of assets. Therefore, the portfolio is less responsive to idiosyncratic shocks and, consequently, the weights of each asset are more stable over time. This, in turn, reduces the cost of rebalancing the portfolio.

This paper follows Magill and Constantinides (1976), Davis and Norman (1990), Leland (2000), Ormos (2011), and investigates the multi-period problem of portfolio management. An investor usually constructs his portfolio at time t, and then the market conditions are changed over time that leads to the fact that the existing portfolio becomes sub-optimal at time $t+\varepsilon$, where ε is a short period. Therefore, there is a need to revise the portfolio over time.

Investors can adopt different rebalancing strategies. Some people rebalance the portfolio only at a certain time ("time-only"), some – only if a certain threshold is reached ("threshold-only"). The essence of the latter is that if the difference between old and new weights exceeds a certain threshold, then the investor rebalances his portfolio. Furthermore, some investors combine these two rebalancing strategies ("time-and-threshold"). This thesis uses the "time-and-threshold" rebalancing strategy.

Different researchers also apply different rebalancing frequencies: Leland (2000), Donohue (2003), Kritzman (2009) – annual, quarterly; Lynch (2010), Malamud (2014) – monthly; Gârleanu (2009), Zakamulin (2016) – weekly; Ormos (2011), Ha (2017) – daily. We follow Gârleanu (2009) and Zakamulin (2016), and use weekly rebalancing frequency to maintain optimal risk-and-return characteristics. Jaconetti (2010) argues that 5% threshold is the optimal value for portfolio rebalancing, so we choose this threshold value for our analysis as well.

In this thesis, we assume that transaction costs, i.e. brokerage fees, are proportional and equal to 10 basis points of the traded volume both for sale and purchase in a similar fashion to Ormos (2011) and Ha (2017). The lower are the costs associated with managing the portfolio, the better off the investor is. For performing the comparative analysis of HRP and MV we use the following metrics: the Sharpe ratio, the standard deviation of portfolio returns, cumulative transaction costs, the share of the portfolio that should be rebalanced to maintain the optimal allocation of assets within the portfolio.

In addition, following Gârleanu (2009), Ormos (2011), Alipour (2016) this thesis does not take into account short sales with the purpose of simplifying the comparative analysis. It is assumed that an investor does not intend to withdraw all capital and close down all positions until the end of the period, which will be considered here. Furthermore, the investor does not pay taxes on portfolio capital gains over the whole period in a similar fashion to Kritzman (2009), Gârleanu (2009) and Ormos (2011). However, short sales and taxes can be easily added to the models.

We will also assume that the investor is risk-averse and wants to minimize the risk of the portfolio at a given level of portfolio return, as in Markowitz (1952).

The main question of this thesis can be formulated in the following way: "Can a more stable portfolio provided by HRP be more efficient than that provided by MV under different market conditions?" The more stable portfolio is the one that is rebalanced fewer times than others. Stability is also important because the investor can decrease the total value of transaction costs, and, as a result, increase portfolio return. However, there can be a situation when even a very unstable portfolio can outperform the stable one.

This paper is organized as follows: Chapter 2 provides us with the literature review of main findings related to the construction of the diversified portfolio; Chapter 3 describes the methodology of different portfolio diversification approaches and the main metrics used for making the comparative analysis; Chapter 4 provides us with descriptive statistics of the data; Chapter 5 presents the empirical results; the conclusions are made in Chapter 6.

Chapter 2

LITERATURE REVIEW

Markowitz (1952) describes the process of selecting optimal portfolio in a single-period model. In accordance with his view, the investor is willing to get a high return, and at the same time, he is against a high volatility of return. This paper describes how the investor can estimate the expected return of portfolio using a general statistical formula called the expected value; the corresponding risk can be measured by the standard deviation. In order to take into account the correlation between securities, the author suggests calculating a covariance value. These parameters can be estimated using historical data on available securities.

The model can be formulated in terms of a set of equations solving which the investor can find the optimal weights for each security in his portfolio. In addition, Markowitz (1956) imposes non-negativity restrictions on these weights. Overall, the author describes how to construct the portfolio from the chosen set of securities without taking into account transaction costs. The Markowitz Mean Variance (MV) approach is used as one of two methods for performing the comparative analysis in this thesis.

Pogue (1970) describes how variable transactions' costs, short sales, leverage policies and taxes influence the selection of portfolio. This paper considers two types of transactions' costs: volume related price effects and the brokerage fees. The transaction costs depend on the number of shares purchased or sold. In addition, the author describes how the investor can include capital gains and dividend income taxes in his portfolio. This paper provides us with the portfolio selection model in the single period with transaction costs and other market imperfections. Overall, if the investor considers transaction costs during portfolio construction, the efficient frontier narrows.

The problem of portfolio construction in the presence of transaction costs has also been considered by Constantinides and Magill (1976), Constantinides (1986), Davis and Norman (1990), Shreve and Soner (1994), Dumas and Luciano (1991), Leland (2000), Pliska and Suzuki (2004), Ormos (2011), Zdorovenin (2012), Muhle-Karbe (2016) and many others.

Sharpe (1966, 1994) presents the concept of reward-to-variability ratio. In fact, this ratio called Sharpe ratio and measures the expected excess return per unit of the expected risk. The author suggests using this ratio to choose efficient assets from the set of all available financial assets. In fact, the investor can allocate funds among the set of available funds maximizing the Sharpe ratio of his portfolio. This ratio is used as one of the crucial indicators of portfolio performance in this thesis.

Zakamulin (2016) investigates the multi-period portfolio optimization problem over the years from 1989 till 2014. He uses S&P500 for constructing the portfolio and applies GARCH and EWMA approaches. Zakamulin (2016) utilizes daily, weekly and monthly rebalancing frequencies and uses the annualized 2% risk-free rate. We follow Zakamulin (2016) and utilize the same risk-free rate of return but for the period from 1989 to 2017.

Clarke (2006), overall, focuses on the comparison of two portfolios: the market one, which is the capitalization-weighted portfolio containing 1000 US stocks, and the minimum-variance portfolio. In order to solve the non-invertability problem of the covariance matrix, the author applies the asymptotic principal components procedure suggested by Connor and Korajczyk (1988), and the Bayesian shrinkage procedure of Ledoit and Wolf (2004). As a result, he obtains two minimum-variance portfolios. The author considers the years from 1968 till 2005, assuming that the T-bill return was 5.9% over that period on average, applying the annual rebalancing strategy. He argues that if the investor looks back at a one-year rolling window of daily returns instead of a five-year rolling window of monthly returns it enhances the accuracy of the estimated covariance matrix. Gosier et al. (2005) make a similar suggestion.

Clarke (2006) shows that the minimum-variance portfolio delivers lower the portfolio risk and almost the same portfolio return as the market one.

Jaconetti (2010) presents the best practices for portfolio rebalancing. The author describes the impact of the frequency of rebalancing on the portfolio performance. There are three main rebalancing strategies such as "time-only", "threshold-only", and "time-and-threshold".

The first strategy takes into account only time. The investor rebalances portfolio at regular intervals - monthly, quarterly, semiannually or annually depending on his choice.

If the investor chooses the second rebalancing strategy, then his portfolio is rebalanced only when the certain threshold is reached. The threshold is the deviation from its target allocation of assets within a portfolio.

The third strategy is more complicated because the investor revises his portfolio only at the certain time frequency and as long as the minimum threshold such as 1%, 5%, 10% is reached. Overall, if a portfolio rebalanced monthly or quarterly, then the number of rebalancing events and the corresponding costs increase significantly relatively to the portfolio that is rebalanced semiannually or annually. Jaconetti (2010) concludes that annual or semiannual rebalancing with 5%

thresholds minimizes costs and provides a good balance between risk and control.

The annual rebalancing frequency is quite common to the literature as, for instance, in Clarke (2006), Behr et al. (2013), Chow et al. (2013), Chow et al (2016), and Bielstein (2017). However, we use weekly rebalancing strategy as in Gârleanu (2009), Zakamulin (2016), with 5% threshold to maintain optimal risk-and-return characteristics of the portfolio and to lower average deviations from the desired assets weights in the portfolio. In addition, we will make the sensitivity analysis to show how different thresholds influence the portfolio performance provided by MV and HRP approaches.

Marasovic (2015) outlines the impact of transaction costs on an investment portfolio considering the Croatian capital market. The author applies the extended Markowitz approach developed by Mitchell (2002). Mitchell (2002) explains in his paper how the investor can include the transaction costs when he constructs the efficient frontier. Marasovic (2015) considers symmetric brokerage fees for buying and selling of stocks: the case of 1.25% and 0.35% proportional transaction fees are analyzed.

Overall, researchers utilize different proportional brokerage fees: Leland (2000), Donohue (2003), Ormos (2011), Ha (2017) – 0.10%; Baule (2008) – 0.25%; Hautsch (2017) – 0.5%; Kritzman (2009) – 0.4% - 0.75%, and others. We follow Leland (2000), Donohue (2003), Ormos (2011) and Ha (2017) and choose proportional transaction fees that equal to 0.1% of the traded volume both for sale and purchase. Since the correlation between the volume of transactions and expected portfolio return is negative, the efficient frontier obtained after rebalancing and incurring brokerage fees is positioned below the efficient frontier that the investor draws before rebalancing.

Lopez de Prado (2016) describes the hierarchical risk parity (HRP) method that applies modern mathematics (graph theory and machine learning techniques) in order to build a diversified portfolio based on the information contained in the covariance matrix. This approach delivers lower out-of-sample portfolio variance than MV. In addition, HRP does not require the invertibility of the covariance matrix. The author emphasizes the following problems of Markowitz's MV approach that HRP can avoid: instability, concentration, and underperformance.

Lopez de Prado (2016) simulates the data and then shows that MV is unstable in response to the idiosyncratic and common shocks. Overall, the main contribution of this paper is a new portfolio diversification approach that provides more stable portfolio over time. It means that the investor can rebalance his portfolio fewer times in comparison to MV and incur lower transaction costs. In addition, the author argues that HRP provides a higher Sharpe ratio and a lower volatility out-of-sample than MV, even if it is the main goal of the Markowitz's approach. The HRP approach is used as the second method for performing the comparative analysis in this thesis.

Alipour (2016) presents a quantum-inspired hierarchical risk parity approach (QHRP) that solves the optimization problem of portfolio diversification using a quantum annealer. Quantum annealing is the process of optimization problem that can be encoded as a Hamiltonian. The author claims that minimum-variance portfolio optimization method provides less reliable solutions and the main reason is the inversion of a covariance matrix. However, a simple risk parity approach ensures stable portfolios. Mainly, this paper is based on the conclusions of Lopez de Prado (2016).

In addition, Alipour (2016) suggests solving a quadratic unconstrained binary optimization problem to minimize the quasi-block-diagonalization objective function. Afterward, the recursive bisection problem is solved to find the weights for each asset in the portfolio. Overall, the author compares different diversification approaches such as the inverse-variance parity (IVP), HRP, mean-variance (MV), and QHRP by the annualized volatility, the average price of the portfolio over time, the Sharpe ratio and others. For this purpose, thirty-eight futures contracts (on stocks, bonds, commodities) were used for the first testing of portfolio performance and thirty stocks of the Dow Jones Industrial Average (DJIA) for additional testing. The author also suggests looking back at a one-year rolling window of daily returns at each rebalancing date. We follow this suggestion in our methodology. The same rolling window is used by Gosier et al. (2005) and Clarke (2006).

In the case of utilizing thirty-eight futures contracts, the average price of the portfolio and the annualized volatility provided by MV is the highest; meanwhile, QHRP has the lowest annualized volatility and the highest average price of the portfolio among other methods. In addition, the Sharpe ratio provided by MV is 0.51, while this value for QHRP is 0.82. When the author uses thirty-eight futures contracts QHRP outperforms MV. However, in the case of utilizing of thirty stocks of the DJIA, the annualized volatility provided by MV and QHRP are almost the same. It is explained by the little hierarchical structure in the dataset. In other words, all thirty stocks are very correlated with each other, as a result, it is hard to single out clusters of assets. Moreover, the covariance matrix has low condition number that enhances the stability of two portfolios. Overall, QHRP can provide the lower annualized volatility in comparison to MV, but the performance of the last one can be similar, depending on the dataset quality.

Bielstein (2017) suggests using the implied cost of capital as an expected return proxy to construct the Markowitz portfolio in the same manner as it is described by Gebhardt et al. (2001). This is consistent with previous studies published by Nekrasov and Ogneva (2011) and Mohanram and Gode (2013). Bielstein (2017) follows Fama and French (1993), Chow et al. (2011), Novy-Marx and Velikov (2014), and includes only common stocks listed in the NYSE, AMEX, and NASDAQ. In addition, Bielstein (2017) imposes an upper bound of 5% for each asset in the portfolio, follows Novy-Marx and Velikov (2014), and assumes trading costs of 50 basis points, and chooses a yearly rebalancing cycle as well as in many other works of Behr et al. (2013), Chow et al. (2013), Chow et al (2016). Bielstein (2017), considers the entire period from 1985 to 2015 that is divided into six sub-periods with a duration of 5 years to analyze the sensitivity of his results.

To conclude, we considered different portfolio diversification approaches (MV, HRP, QHRP, IVP), their advantages, the papers that outline the effect of transaction costs on the efficient frontier, and, as a result, on the portfolio performance. We also defined the basic strategies of portfolio rebalancing. Further, the findings of these papers will be used to compare the performance of portfolios provided by MV and HRP for thirty stocks of DJIA for the period from 1990 to 2017.

Chapter 3

METHODOLOGY

This section contains three parts. The first one describes the minimum-variance portfolio diversification method including the transaction costs. The second one provides us with the general description of the HRP approach including brokerage fees too. The last one contains all assumptions and criteria used for performing the comparative analysis of MV and HRP.

3.1. Description of MV approach

In accordance with the Markowitz's mean-variance approach, the investor chooses minimum risk at a given level of return. Consequently, the investor can draw the set of portfolios lying on the efficient frontier.

This paper utilizes the extended MV approach that includes the transaction costs described by Mitchell (2002).

Overall, the investor should solve the optimization problem, where the first three conditions (1)-(3) represent the standard portfolio problem. We denote $\boldsymbol{w}$ as the vector of weights, $\boldsymbol{e}$ as the vector of ones, $\boldsymbol{C}$ as the covariance matrix, $\boldsymbol{E(R)}$ as the column vector of expected returns, and $\boldsymbol{E}$ as the desired expected portfolio return.

The next three conditions (4)-(6) represent the rebalancing problem, where $\overline{\boldsymbol{w}}$ is the vector of new weights of a new efficient portfolio. We denote $\boldsymbol{b}$ and $\boldsymbol{s}$ as the vectors of the amount bought and sold of the corresponding security and denote $\boldsymbol{C_B}$ and $\boldsymbol{C_S}$ as the vectors of transaction costs that the investor incurs when he buys and sells one unit of the corresponding security, respectively. In addition, we assume the proportional brokerage fees. Thus, the optimization problem looks as follows:

$$\min \left\{ w^T \cdot C \cdot w \right\} \tag{1}$$

$$\text{Subject to: } E(R)^T \cdot w \geq E \tag{2}$$

$$e^T \cdot w = 1 \tag{3}$$

$$\overline{w} - b + s = w \tag{4}$$

$$\left(C_B + e\right)^T \cdot b + \left(C_S + e\right)^T \cdot s = 0 \tag{5}$$

$$b, s, w \geq 0. \tag{6}$$

Overall, this paper suggests constructing the portfolio in the initial period $\boldsymbol{t}$ solving the basic optimization problem (1)-(3) in such a way the efficient frontier could be constructed. After that, it is proposed to find the highest value of the Sharpe ratio in order to find the tangency portfolio. The weights of securities are nonnegative. Short sales are not allowed as in Gârleanu (2009), Ormos (2011) and Alipour (2016). The investor chooses the tangency portfolio that lies on the efficient frontier in each period. We assume that there are no taxes on portfolio capital gains over the whole period. However, short sales and taxes can be easily added to the models.

In the next period $\boldsymbol{t+\varepsilon}$, where $\boldsymbol{\varepsilon}$ is a short period, the investor should solve the extended optimization problem (1)-(6) in order to identify a new efficient frontier. Then, he finds again the portfolio with the highest Sharpe ratio and chooses it. This procedure is repeated as many times as there are rebalancing periods.

3.2. Description of HRP approach

The hierarchical risk parity approach is the modern portfolio diversification method developed by Lopez de Prado in 2016. This one can construct a diversified portfolio even using a singular covariance matrix that is impossible for MV method. There can be highlighted the stages of this approach such as tree clustering, quasi-diagonalization, and recursive bisection.

Let us consider in a more detail the first stage. Firstly, we assume that we have $\boldsymbol{N}$ assets with returns series over time $\boldsymbol{T}$. Consequently, we can construct the corresponding correlation and covariance matrices. After that, we should apply a hierarchical clustering algorithm based on the Euclidean distance metric. It is a basic clustering method in the set of machine learning techniques that enables us to single out the clusters of assets and find the cluster order that is used to transform a general covariance matrix to the diagonal one. Overall, the outcome of this stage is the cluster dendrogram (Appendix A).

The aim of the second stage is the reorganization of rows and columns of the covariance matrix using the order of clusters obtained in the previous stage. As a result, the largest values lie along the diagonal. Overall, the investor transforms a general covariance matrix to diagonal form using the cluster order (Appendix A). This procedure lowers the condition number, as a result, the portfolio becomes more stable over time.

The last stage is the recursive bisection represented in the same fashion as described by Lopez de Prado (2016). The recursive bisection, basically, based on a simple rule: the higher is the variance of stock, the lower is the corresponding weight in a portfolio (Appendix B).

This stage guarantees that all weights are higher than zero and sum up to one. This method gives us the only portfolio that is optimal. Therefore, the rebalancing problem is applied only to one portfolio that is solved in a similar way as for MV in each period.

3.3. Main criteria for performing the comparative analysis

This section describes the main assumptions, the criteria for performing the comparative analysis such as the price of the portfolio, the annualized volatility, the Sharpe ratio, and the cumulative costs of portfolio rebalancing.

This paper takes into account the set of assumptions:

1. The investor wants to minimize the risk of the portfolio at a given level of portfolio return. The investor is the risk-averse. This is standard to the literature, starting from Markowitz (1952).

2. The investor constructs his portfolio in the first period and then manages it over time. We follow Gârleanu (2009) and Zakamulin (2016), and use weekly rebalancing frequency to maintain optimal risk-and-return characteristics of the portfolio. The threshold is set as 5% as Jaconetti (2010) suggests.

3. This thesis does not take into account short sales as in Gârleanu (2009), Ormos (2011), Alipour (2016). Furthermore, the investor does not pay taxes on portfolio capital gains over the whole period.

4. The investor holds his portfolio to the terminal date.

5. The brokerage fees are proportional and equal to 10 basis points in a similar fashion to Leland (2000), Ormos (2011), Ha (2017).

6. The investor looks back at a one-year rolling window of daily returns to construct a new portfolio at each rebalancing date in the same manner as in Gosier et al. (2005), and Clarke (2006), Alipour (2016).

The rate of return of each asset in the period t is estimated as follows:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{7}$$

We assume that the investor is interested in low portfolio risk and low transaction costs. To evaluate portfolio performance, we use the indicators such as the Sharpe ratio, the annualized volatility, the portfolio price, and the cumulative costs of portfolio rebalancing.

We follow Ledoit and Wolf (2008) and apply a studentized circular bootstrap approach developed by Politis and Romano (1994) to test the significance of the difference between the Sharpe ratio of MV and HRP (Appendix C). In addition, we follow Ledoit and Wolf (2011) to test the significance of the difference between the variance of portfolio returns of MV and HRP (Appendix D). We also test the significance of the difference between other indicators of MV and HRP using bootstrapping technique.

The higher is the Sharpe ratio, the more the investor is satisfied. This term is a reward-to-variability measure estimated as described by Sharpe (1966, 1994):

$$SR = \frac{E(R_p) - r_f}{\sigma_p}, \tag{8}$$

where $E(R_p)$ is the expected portfolio return, r_f - is the risk-free rate, σ_p is the standard deviation of the portfolio. In fact, this ratio measures the expected excess return per unit of the expected risk. We follow Zakamulin (2016) and utilize the annualized 2% risk-free rate over the years from 1989 to 2017.

The lower the annualized volatility of returns is, the more the investor is satisfied. This indicator is a simple standard deviation of returns during one year estimated by the following formula:

$$SD_t = \sqrt{\frac{\sum_{i=1}^{n}(r_i - \bar{r})^2}{n-1}}, \tag{9}$$

where ***t*** stands for year, ***n*** is the number of periods within one year, r_i is the actual portfolio return at the end of the period ***i*** within the year ***t***.

The asset manager tries to incur transaction costs as low as possible. The cumulative costs of portfolio rebalancing are just a sum of all brokerage costs incurring over a given period. The following formula is used:

$$CC_t = \sum_{i=1}^{n} TC_i, \tag{10}$$

where ***t*** stands for a given period, TC_i is the transaction cost incurred over a given period ***i*** within the whole period ***t***.

Chapter 4

DATA DESCRIPTION

We follow Ormos (2011), Alipour (2016) and choose thirty stocks of the Dow Jones Industrial Average (DJIA) for our analysis. DJIA is a stock market index listed on the NYSE and NASDAQ, and it covers companies only with a large capitalization and high liquidity. We can highlight some of them: Apple, Boeing, Coca-Cola, IBM, Microsoft, etc. The full list of companies included in the index can be found in Appendix E.

The time interval considered in this thesis is 03/01/1989 – 01/01/2018. This period covers the US business cycle expansions and contractions as defined by the National Bureau of Economic Research[1]. This enables us to analyze the sensitivity of our results to different economic periods. Therefore, the whole period is split into four sub-periods (Appendix F): January 1990 – November 2001, November 2001 – December 2007, December 2007 – June 2009, and June 2009 – December 2017.

One complication for our analysis comes from the fact that the composition of the DJIA is not constant over the whole period (Appendix E). We have to take this into account when constructing the portfolio. Overall, the portfolio is constructed only from those stocks that constitute the DJIA at the rebalancing date in a similar fashion to Alipour (2016).

The total number of trading days during the whole period is 7308. The first date is 03/01/1989, the last date – 29/12/2017. This thesis suggests rebalancing the

[1] http://www.nber.org/cycles.html.

portfolio on the first business day of each week, and the investor should look back at a one-year rolling window of daily returns in a similar way as in Gosier et al. (2005), and Clarke (2006), Alipour (2016).

Initially, the investor constructs his portfolio on 01/01/1990. Thus, the first rebalancing date is 08/01/1990, and the last – 01/01/2018. The total number of portfolio rebalancing dates is 1461.

We follow Zakamulin (2016) and utilize the annualized 2% risk-free rate over the years from 1989 to 2017.

Statistical description of stocks taking dynamic portfolio composition into account over the entire period is outlined in Table 1. The minimum number of trading days when the stock is in the portfolio is 591 (Navistar or NAV), the maximum – 7308, the average - 5209. Table 1 contains statistical characteristics such as average annual return, standard deviation, skewness, kurtosis, and the Sharpe ratio.

Table 1. The statistical description of all stocks over the whole period

Stock	Number of trading days	Average annual return, %	Standard deviation, %	Skewness	Kurtosis	Sharpe ratio
MMM	7307	11.9	22.4	0.08	4.97	0.53
AA	6231	7.9	37.9	0.29	8.53	0.21
MO	4822	15.7	29.6	-0.71	16.0	0.53
AXP	7308	15.2	34.7	0.33	8.63	0.44
AIG	4972	5.8	36.8	-4.12	165	0.16
AAPL	706	13.4	22.5	-0.08	3.47	0.60
T	6602	8.1	26.3	0.23	5.62	0.31
BAC	1409	8.7	75.3	0.73	11.8	0.12
CAT	7308	15.2	31.4	0.09	4.39	0.48

Table 1 continued

Stock	Number of trading days	Average annual return, %	Standard deviation, %	Skewness	Kurtosis	Sharpe ratio
CVX	7308	11.2	24.5	0.29	9.94	0.46
CSCO	2158	10.8	25.0	-0.37	20.0	0.43
C	5150	15.1	48.6	1.51	48.8	0.31
KO	7308	12.3	22.4	0.20	6.08	0.55
DWDP	7308	9.5	31.5	0.02	7.37	0.30
XOM	7308	9.7	23.0	0.28	9.27	0.42
GE	7308	9.2	27.6	0.29	9.19	0.33
GS	1077	12.5	21.5	-0.29	1.85	0.58
GT	2737	8.7	29.1	0.13	2.66	0.30
HPQ	6231	12.5	39.3	0.07	6.46	0.32
HD	4571	12.2	31.3	-0.38	14.8	0.39
HON	4822	15.1	32.1	0.30	14.4	0.47
IBM	7308	9.3	27.3	0.22	7.57	0.34
INTC	4571	8.1	37.4	-0.10	7.88	0.22
IP	3848	8.3	29.6	0.30	2.84	0.28
JNJ	7308	13.6	21.4	0.03	6.62	0.64
JPM	6717	17.6	37.8	0.75	13.1	0.47
MDLZ	1010	8.3	22.3	-0.42	5.72	0.37
MCD	7308	14.6	24.1	0.11	4.84	0.60
MRK	7308	9.8	27.0	-0.53	14.0	0.36
MSFT	7308	24.0	32.4	0.19	6.21	0.74
NAV	591	-2.2	56.0	0.45	2.22	-0.04
NKE	1077	16.3	21.8	1.16	11.8	0.75
PFE	7308	12.9	26.7	-0.01	3.51	0.48
PG	7308	12.4	22.2	-1.39	34.3	0.56
TRV	7308	12.4	27.1	0.73	17.1	0.46
UNH	1326	28.1	19.9	0.23	2.70	1.41
UTX	7308	14.5	25.9	-0.38	13.1	0.56
VZ	3460	5.4	20.6	0.48	9.68	0.26
V	1077	21.7	19.8	0.48	6.57	1.10
WMT	7308	14.7	26.2	0.24	4.15	0.56
DIS	7308	14.6	29.4	0.23	7.56	0.50
BA	7308	15.1	29.5	-0.10	6.19	0.51

The minimum average annual return of the share is -2.2% of NAV stock (Navistar), the maximum value is 28.1% of UNH stock (United Health Group), the average value across all stocks is – 12.4%. Even though the whole period covers two economic crises, the stocks have positive average annual returns on average. The standard deviations of annual returns are in the range 19.8% – 75.3%. The riskiest stock is BAC (Bank of America Corporation) with the value of the standard deviation of 75.3%, but the most riskless one is UNH (19.8%). Skewness values are in the range - -4.12 – 1.51, kurtosis values are in the range 1.85 – 165. It means that distributions of annual returns of all stocks are non-normal. The Sharpe ratios are in the range - -0.04 – 1.41. The NAV stock has the minimum Sharpe ratio, the UNH stock has the maximum one, the average value is 0.46. Overall, all shares have positive Sharpe ratio when they are in the portfolio.

In addition, statistical description of stocks taking dynamic portfolio composition into account for each particular sub-period is outlined in Appendix G.

Chapter 5

EMPIRICAL RESULTS

First, we conduct the comparative analysis of MV and HRP approaches for the whole period from 1990 till 2017. Afterward, the entire period is split into four sub-periods: January 1990 – November 2001, November 2001 – December 2007, December 2007 – June 2009, and June 2009 – December 2017 in order to check the behavior of portfolio diversification models in various economic states.

This thesis uses the "time-and-threshold" rebalancing strategy. We assume that the investor revises his portfolio each week with 5% threshold. If the investor uses the MV approach then the average actual annual portfolio return is 9.90%, while this value for the second method is 9.87%. Thus, the two approaches provide almost the same annual portfolio return during the period 1990 - 2017 for thirty stocks of the DJIA. The portfolio risk (volatility) of MV portfolio over the whole period is 18.7%, meanwhile, this value for HRP portfolio is 14.6%. The Sharpe ratios for the entire period are 0.42 and 0.54 for MV and HRP models, respectively.

As described in the methodology section, we follow Ledoit and Wolf (2008) and apply a studentized stationary circular bootstrap approach developed by Politis and Romano (1994) to test the significance of the difference between the Sharpe ratios of the MV and HRP approaches. We use heteroskedasticity and autocorrelation robust (HAC) kernel estimation in a similar way as in Andrews and Monahan (1992), Ledoit and Wolf (2008) to check autocorrelation of the returns. We use VAR(1) model in conjuction with bootstrapping the residuals.

The number of bootstrap replications for computing the p-value is 5000, the block length in the circular bootstrap is 5, we construct a two-sided confidence interval to calculate the symmetric p-value. This is standard to the literature, starting from Ledoit and Wolf (2008). The null hypothesis is that the difference between the Sharpe ratio of MV and HRP is equal to zero. We obtained that t-statistic and p-value are -1.12 and 0.26, respectively. As a result, we fail to reject the null hypothesis at 90% confidence interval, so the values of the Sharpe ratio of MV and HRP are not statistically different.

In addition, we follow Ledoit and Wolf (2011) and apply a studentized stationary circular bootstrap approach developed by Politis and Romano (1994) to test the significance of the difference between the variance of portfolio returns of MV and HRP. We test the following null hypothesis: the difference between the variance of portfolio returns of MV and HRP is equal to zero. We obtained that the p-value of this test is 0.001, so we reject the null hypothesis: this difference is statistically significant. Therefore, we can conclude that the HRP approach produces less volatile portfolio returns than the MV one.

Figure 1 shows the portfolio price generated by the MV and HRP for the period from 1990 till 2017. The initial price is $100 000. We can see that the values of both portfolios increased over the whole period. The price of MV portfolio rose considerably from 1990 to 1999 and reached $742 318, in the meantime, the price of HRP portfolio was only $509 677. MV and HRP portfolio prices fell by 54% and 13%, respectively, in the following years from 1999 to 2003. Afterwards, we can see that the second method produced larger increase in portfolio value than the first one.

During the period from 2003 until the beginning of the financial crisis in 2007, the prices of these two portfolios rose by the factor of 1.5. Later, in 2007-2009, the situation reversed: HRP portfolio price fell more dramatically than MV. Their corresponding returns were -47% and -31%. Since 2009, both portfolios have grown considerably: the price of MV and HRP portfolios rose 2.7 and 3.4 times, respectively. The end price of MV portfolio is $984 273, meanwhile, this value for the other one is $1 181 374. Overall, the price of HRP portfolio was higher than MV portfolio in 60.2% cases.

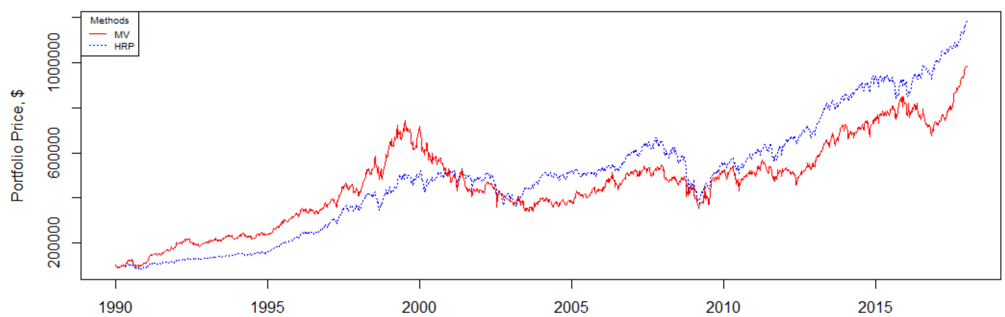

Figure 1. The price of portfolios constructed using MV and HRP, $

In order to define what method is superior, let us analyze the extra indicators such as the transaction costs, the shares of the portfolio that should be rebalanced to maintain the optimal allocation of assets within the portfolio, the annualized Sharpe ratios, and the annualized volatilities of portfolio returns.

The cumulative transaction costs during the whole period are represented in Appendix H. We can see that for the MV approach this value reached $217 394 at the end of 2017. For the HRP approach, cumulative transaction costs amounted to only $60 144, which is 3.6 times lower than for MV. In addition, we test the difference between the transaction costs of MV and HRP using the bootstrapping technique with 1000 replications. As a result, we obtained the value of t-statistic that equals 36.64. Consequently, this difference is statistically significant so the Markowitz approach is more costly; therefore, it is optimal to choose HRP from the cost minimization point of view.

The shares of the portfolio that should be rebalanced to maintain the optimal allocation of assets within the portfolio at the first business day of each week presented in Appendix H. The average portfolio weight to be rebalanced according to the HRP method is 6.9% and is 2.5 times lower than for MV. Notice that HRP portfolio is kept unchanged 576 out of 1461 times (39.4% of all rebalancing cases), while the Markowitz portfolio is not revised only 82 out of 1461 times (5.6% of all rebalancing cases) when the threshold is 5%. Overall, the HRP approach is 7 times more stable than the classical one. Thus, the investor can rebalance his portfolio fewer times, as a result, he can incur lower transaction costs.

Moreover, if the threshold is increased to 10% then HRP and MV portfolios will be rebalanced approximately only in 9.45% and 78.6% cases, respectively. In this case, the average annual return of the HRP portfolio will be 10.24%, while this value will be 10.07% for the MV one.

In addition, we test the difference between the share of the portfolio that should be rebalanced provided by MV and HRP using the bootstrapping technique. As a result, we obtained the value of t-statistic that equals 36.60. Consequently, this difference is statistically significant. To conclude, the HRP portfolio diversification method is much more stable than the MV one.

We consider the rolling annualized volatility of annual portfolio returns and the rolling annualized Sharpe ratio for validation of our findings made above.

The rolling annualized volatilities of annual portfolio returns provided by MV and HRP are shown in Appendix H in a similar fashion as in Clarke (2006). Overall, we can see that the HRP portfolio is less risky over the whole period except for 2008. Moreover, we test the significance of the difference between the rolling annualized volatility of annual portfolio returns of MV and HRP in the same way as in Ledoit and Wolf (2011). We obtained the p-value that equals 0.059[2]. Consequently, we can say that the difference is statistically significant at 90% confidence interval. Therefore, we can say that the HPR portfolio is less risky.

The last portfolio performance indicator used in the comparative analysis is the rolling annualized Sharpe ratio. It is one of the most common measures used for comparing different portfolio diversification approaches. The annualized Sharpe ratios of MV and HRP models are represented in Appendix H. Overall, this value is higher for HRP in 66.1% cases of the entire period. We test the significance of the difference between the rolling annualized Sharpe ratios of MV and HRP in a similar way as in Ledoit and Wolf (2008). We obtained the p-value that equals 0.43[3]. Consequently, we can say that the difference is not statistically significant at 90% confidence interval. This suggests that the HRP and MV portfolios provide the same Sharpe ratio.

We have considered so far only the empirical results for the entire period. Let us look separately into the four sub-periods in more detail. It is supposed that the results obtained for the whole period consistent with the results in each particular market state.

Table 2 presents four sub-periods in which MV and HRP are compared by the average annual portfolio return, the standard deviation of the portfolio returns,

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[2] This value is computed for every 52 weeks separately. The average p-value is 0.059, minimum p-value is 0.01, maximum p-value is 1.0. The brown line represents the periods when the difference between the annualized volatility of MV and HRP is statistically significant at 90% confidence interval (Appendix H).

[3] This value is computed for every 52 weeks separately. The average p-value is 0.43, minimum p-value is 0.006, maximum p-value is 0.99. The brown line in figure 6 represents the periods when the difference between the annualized Sharpe ratio of MV and HRP is statistically significant at 90% confidence interval (Appendix H).

the Sharpe ratios, the compound annual growth rates (CAGR), the cumulative transaction costs, the average portfolio weights to be rebalanced, and rebalancing frequency[4].

Table 2. The comparative analysis of four sub-periods

Indicator	Jan 1990 – Nov 2001		Nov 2001 – Dec 2007		Dec 2007 – June 2009		June 2009 – Dec 2017	
	MV	HRP	MV	HRP	MV	HRP	MV	HRP
Average annual return, %	14.3	14.4	4.8	5.4	-10.1	-18.9	11.8	11.5
Standard deviation, %	19.5	14.7*	16.8	13.1*	30.0	28.5	16.2	11.6*
Sharpe ratio	0.63	0.84	0.16	0.26	-0.40	-0.73	0.60	0.82
CAGR, %	0.24	0.26	0.07	0.09	-0.28	-0.44	0.20	0.21
Cumulative transaction costs, $	68K	16K*	10K	3K*	2K	0.6K*	24K	6K*
Average portfolio weight to be rebalanced, %	15.6	7.2*	17.2	6.8*	27.6	6.9*	17.3	6.5*
Rebalancing frequency, %	98.7	62.2	92.1	62.0	64.1	69.2	95.5	55.5

Let us consider the first sub-period January 1990 – November 2001 that is considered as the period of economic expansion and a brief recession at the end of the period. The HRP portfolio has a higher average annual portfolio return and a lower standard deviation, and, consequently, a higher Sharpe ratio. In addition, CAGR is larger for HRP approach than for MV.

[4] This value is calculated as the ratio between rebalancing cases when the investor actually rebalance his portfolio and all possible rebalancing events over particular sub-period. For example, if the sub-period contains 100 possible rebalancing events, and the investor revises his portfolio only in 30 cases out of 100 than the rebalancing frequency is 30%.

On the other hand, the rebalancing costs of the Markowitz portfolio is 4.24 times higher. The average share of the portfolio to be rebalanced for MV portfolio is 15.6, that is 2.2 times higher than for HRP. It is associated with the fact that the Markowitz portfolio is revised in 98.7% cases and it is 1.6 times more frequent than for the other portfolio. Overall, the HRP approach seems to perform better than MV in this sub-period[5]. The same situation is observed in the sub-periods November 2001 – December 2007 (the period of economic expansion), and June 2009 – December 2017 (the period of post-recession recovery).

However, the totally contrary results are obtained for the period of December 2007 – June 2009 (the period of financial crisis). In Table 2 we can see that the MV portfolio has a higher average annual portfolio return, Sharpe ratio, and CAGR, but it is also more expensive approach. The MV approach mainly contains the stocks MCD (McDonald's) and WMT (Wal-Mart), which were not extremely sensitive to the financial crisis. This portfolio consisted only of a few assets that minimize losses of an investor. In accordance with the concept of HRP approach, an investor should invest his funds in *all* available stocks. Taking into account that the entire market dropped down, it was not the best strategy to follow. The HRP portfolio also contains MCD, WMT stocks but in much lower proportions. As a result, the other equities in the portfolio affected by the crisis led to higher losses of HRP than in the case of MV method.

To conclude, the Markowitz approach deals with the Great Recession better than the HRP method. Even though this sub-period strongly affected the HRP portfolio performance, we can see in Figure 1 that the HRP portfolio has grown faster in the following sub-period. Since we cannot predict when the next

[5] The differences between the standard deviation, cumulative transaction costs, and average portfolio weight to be rebalanced of MV and HRP are statistically significant at 99% confidence interval. It is indicated by * in Table 2.

recession will occur, we can conclude that the HRP approach seems better than MV over the whole period in accordance with our results.

In addition, we make the sensitivity analysis of portfolio performance provided by MV and HRP depending on brokerage fees and thresholds (Appendix I). We use the following transaction fees: 0%, 0.1%, 0.3%, 0.7%, and 1.0%. We utilize the following thresholds: 0%, 5%, and 10%. As a result, we can highlight some relationships between the threshold and each indicator:

1. The larger the threshold is, the larger the standard deviation, the Sharpe ratio, CAGR, the average annual return, the share of the portfolio that should be rebalanced are.

2. The larger the threshold is, the lower the cumulative transaction costs and rebalancing frequency are.

We observe the following interconnections between the transaction fees and each indicator:

1. The larger the transaction fees are, the larger the standard deviation and cumulative transaction costs are.

2. The larger the transaction fees are, the lower the average annual return, CAGR, the Sharpe ratio, and rebalancing frequency are.

The share of the portfolio that should be rebalanced is not affected by the change of brokerage fees.

Notice that rebalancing frequency of MV portfolio is changed from 97.5% to 79% when the threshold is increased from 0% to 10%; in the meantime, this indicator for HRP portfolio is decreased from 100% to 9%. In addition, we test the significance of the difference between the share of the portfolio that should be rebalanced according to MV and HRP using the bootstrapping technique for all 15 cases in the sensitivity analysis. We obtained the p-value that is the same in 15 cases and equals to $2 \cdot 10^{-16}$. Consequently, the difference is statistically significant at 99% confidence interval. Overall, the HRP portfolio is more stable than MV.

We can see in Table 9 (Appendix I) that the standard deviations of the portfolio returns are almost constant over different thresholds and transaction fees. Since we have already tested the significance of the difference between the variance of MV and HRP for one particular case above, then we can conclude that the HRP portfolio is less risky than MV at 99% confidence interval.

There is one interesting fact: if the investor uses the MV approach then the cumulative transaction costs are increased from $0 to $388K when the brokerage fees are increased from 0% to 0.7%. However, when the brokerage fees are 1.0% then the cumulative transaction costs start declining. As a result, the cumulative transaction costs are increased slowly even though the rebalancing frequency is rather high.

We cannot say the same about the HRP portfolio. First, consider the case when there is no threshold and the transaction fees are changed from 0% to 1%. Even though the average share of the portfolio to be rebalanced is 5.3%, the rebalancing frequency is 100%, we can see the rapid increase of cumulative transaction costs in Table 9. Second, consider the case when there is 10% threshold and the transaction fees are changed from 0% to 1%. In this situation, the investor that uses the HRP approach incurs the cumulative transaction costs much lower than in the case described above. This value is increased from $0 to $199K when the brokerage fees are increased from 0% to 1%. At the same time, cumulative transaction costs for MV are changed from $0 to $355K.

This suggests that introducing the 10% threshold positively impacts portfolio performance of the HRP and MV approaches. We tested the significance of the difference between transaction costs that the investor incurs when he uses MV and HRP. This difference is statistically significant at 99% confidence interval.

The last crucial indicator is the Sharpe ratio. We tested the significance of the difference between the Sharpe ratios of MV and HRP for all 15 cases in the sensitivity analysis. As a result, we can make the following conclusion: the difference becomes statistically significant if transaction fees are larger than 0.3%.

Overall, we can conclude that the HRP approach is less risky and less expensive in all cases. The HPR portfolio is more stable if there is the threshold. In addition, this portfolio provides significantly higher Sharpe ratio if transaction fees are larger than 0.3%. Consequently, this method seems better than MV over the years from 1990 to 2017 using 30 stocks of the DJIA except for the period of financial crisis.

Chapter 6

CONCLUSIONS

We compare two portfolio diversification approaches over the period from 1990 to 2017. Overall, we describe portfolio performance over the whole period, over four different sub-periods, and make the sensitivity analysis of the portfolio efficiency depending on brokerage fees and thresholds. We compare the HRP and MV approaches by the Sharpe ratio, the standard deviation of portfolio returns, cumulative transaction costs, and the share of the portfolio to be rebalanced to maintain the optimal allocation of assets within the portfolio. The significance of the difference of a particular indicator of MV and HRP tested by a studentized stationary circular bootstrap approach. The difference between the variance of portfolio returns of HRP and MV is statistically significant at 99% confidence interval for all cases in the sensitivity analysis. We can say the same about the significance of the difference between the share of the portfolio to be rebalanced and cumulative transaction costs provided by two methods. We show that the HRP approach is more attractive for risk-averse investors than MV because it is less risky, more stable, and less expensive. This conclusion is validated using the statistical tools. These two portfolios have almost the same Sharpe ratio if proportional transaction fees are lower than 0.3%. However, the difference is statistically significant at least at 95% confidence interval if proportional transaction fees are higher than 0.3% for cases when the threshold is 0%, 5% or 10%.

The model presented in this paper can be extended. Firstly, we can add short sales and capital gains taxes in the models. Secondly, new indicators for the comparison of methods can be proposed, for instance, the expected tracking error. In addition, a researcher can expand the number of portfolio diversification methods for the comparative analysis, change weekly portfolio rebalancing frequency to monthly, quarterly, semi-annual or annual. Moreover, these models can be tested on the different financial markets.

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APPENDIX A

THE FIRST AND THE SECOND STAGES OF THE HIERARCHICAL RISK PARITY APPROACH

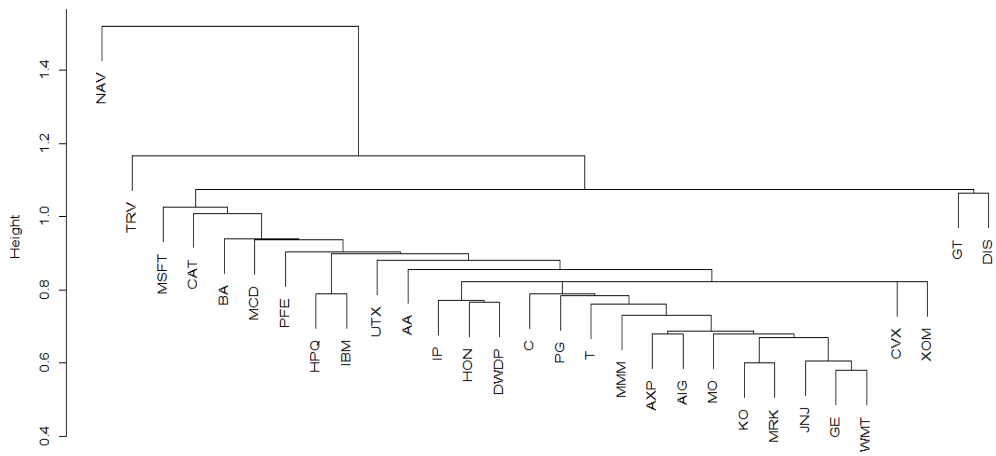

Figure 2. Cluster dendrogram at the first business day

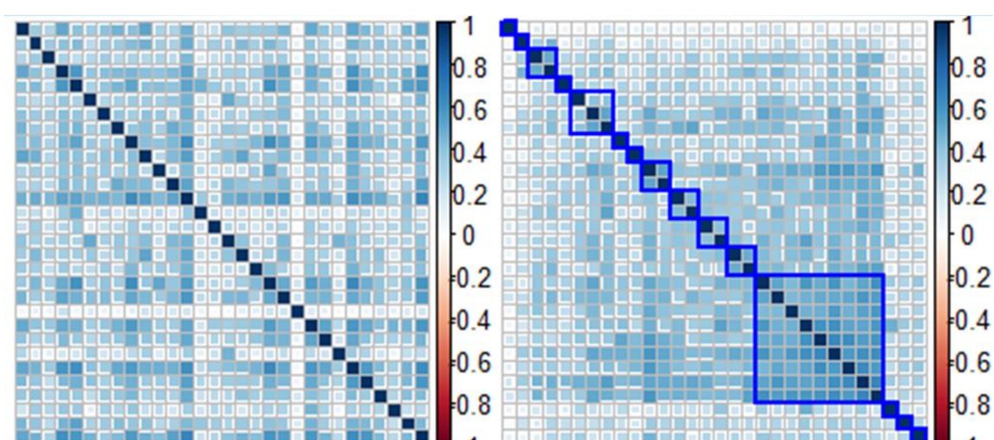

Figure 3. Quazi-diagonalization procedure at the first business day

APPENDIX B

THE RECURSIVE BISECTION STAGE

1. The algorithm is set:

a. Setting the list of items: $L = \{L_0\}$, with $L_0 = \{n\}_{n=1,\dots,N}$

b. Assigning a unit weight to all items: $w_n = 1, \forall n = 1,\dots,N$

2. If $|L_i| = 1, \forall L_i \in L,$ then stop

3. For each $L_i \in L$ such that $|L_i| > 1$:

a. Split L_i into two subsets, $L_i^{(1)} \cup L_i^{(2)} = L_i$, where $\left|L_i^{(1)}\right| = \text{int}\left[\frac{1}{2} \cdot |L_i|\right]$, and the order of clusters is preserved

b. Define the variance of $L_i^{(j)}, j = 1,2,$ as the quadratic form $\tilde{V}_i^{(j)} \equiv \tilde{w}_i^{(j)'} \cdot V_i^{(j)} \cdot \tilde{w}_i^{(j)}$, where $V_i^{(j)}$ is the covariance matrix between the constituents of the $L_i^{(j)}$ bisection, and $\tilde{w}_i^{(j)} = diag\left[V_i^{(j)}\right]^{-1} \cdot \frac{1}{tr\left[diag\left[V_i^{(j)}\right]^{-1}\right]}$, where $diag[.]$ and $tr[.]$ are the diagonal and trace operators

c. Compute the split factor: $\alpha_i = 1 - \frac{V_i^{(1)}}{V_i^{(1)} + V_i^{(2)}}$, so that $0 \le \alpha_i \le 1$

d. Re-scale weights w_n by a factor of $\alpha_i, \forall n \in L_i^{(1)}$

e. Re-scale weights w_n by a factor of $(1 - \alpha_i), \forall n \in L_i^{(2)}$

4. Loop to step two

APPENDIX C

THE TEST OF THE SIGNIFICANCE OF THE DIFFERENCE BETWEEN THE SHARPE RATIO OF MV AND HRP

Define the difference between two Sharpe ratios as follows:

$$\Delta = Sh_i - Sh_n = \frac{\mu_i}{\sigma_i} - \frac{\mu_n}{\sigma_n}, \tag{1}$$

and its estimator is computed as:

$$\hat{\Delta} = \hat{S}h_i - \hat{S}h_n = \frac{\hat{\mu}_i}{\hat{\sigma}_i} - \frac{\hat{\mu}_n}{\hat{\sigma}_n}. \tag{2}$$

We denote the original studentized test statistic by ***d***, that is,

$$d = \frac{\left|\hat{\Delta}\right|}{s\left(\hat{\Delta}\right)}, \tag{3}$$

where $s\left(\hat{\Delta}\right)$ denotes the standard error. We denote the centered studentized statistic computed for the ***m***th bootstrap sample by $\boldsymbol{d^{*,m}}$, $\boldsymbol{m=1,\ldots,M}$, that is,

$$d^{*,m} = \frac{\left|\Delta^{*,m} - \hat{\Delta}\right|}{s\left(\Delta^{*,m}\right)}, \tag{4}$$

where ***M*** is the number of bootstrap resamples. We find a two-sided p-value for the null hypothesis $\boldsymbol{H_0}$: $\Delta=0$. The p-value is computed as follows:

$$PV = \frac{\left\{d^{*,m} \geq d\right\} + 1}{M+1}. \tag{5}$$

APPENDIX D

THE TEST OF THE SIGNIFICANCE OF THE DIFFERENCE BETWEEN THE VARIANCE OF PORTFOLIO RETURNS OF MV AND HRP

Define the difference between two variances as follows:

$$\Delta = \log(\theta) = \log\left(\sigma_i^{\,2}\right) - \log\left(\sigma_n^{\,2}\right), \tag{1}$$

and its estimator is computed as:

$$\hat{\Delta} = \log(\hat{\theta}) = \log\left(\hat{\sigma}_i^{\,2}\right) - \log\left(\hat{\sigma}_n^{\,2}\right). \tag{2}$$

We denote the original studentized test statistic by ***d***, that is,

$$d = \frac{\left|\hat{\Delta}\right|}{s\left(\hat{\Delta}\right)}, \tag{3}$$

where $s\left(\hat{\Delta}\right)$ denotes the standard error. We denote the centered studentized statistic computed for the ***m***th bootstrap sample by $\boldsymbol{d^{*,m}}$, $\boldsymbol{m=1,\ldots,M}$, that is,

$$d^{*,m} = \frac{\left|\Delta^{*,m} - \hat{\Delta}\right|}{s\left(\Delta^{*,m}\right)}, \tag{4}$$

where ***M*** is the number of bootstrap resamples. We find a two-sided p-value for the null hypothesis $\boldsymbol{H_0}$: $\Delta=0$. The p-value is computed as follows:

$$PV = \frac{\left\{d^{*,m} \geq d\right\} + 1}{M + 1}. \tag{5}$$

APPENDIX E

DOW JONES INDUSTRIAL AVERAGE INDEX

Table 3. The description of the DJIA composition

Company	Exchange	Symbol	Period in DJIA	Extended period in DJIA	Period in portfolio
3M	NYSE	MMM	03/01/1989 – 19/02/2008	03/01/1989 – 19/02/2008	03/01/1989 – 29/12/2017
Alcoa	NYSE	AA	03/01/1989 – 26/09/2013	03/01/1989 – 26/09/2013	03/01/1989 – 23/09/2013
Altria Group	NYSE	MO	27/12/2003 – 19/02/2008	03/01/1989 – 19/02/2008	03/01/1989 – 18/02/2008
American Express	NYSE	AXP	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
American International Group	NYSE	AIG	08/04/2004 – 22/09/2008	03/01/1989 - 22/09/2008	03/01/1989 - 22/09/2008
Apple	NASDAQ	AAPL	19/03/2015 – 29/12/2017	19/03/2015 – 29/12/2017	16/03/2015 – 29/12/2017
AT & T	NYSE	T	03/01/1989 – 19/03/2015	03/01/1989 – 19/03/2015	03/01/1989 – 16/03/2015
Bank of America Corporation	NYSE	BAC	19/02/2008 – 26/09/2013	19/02/2008 – 26/09/2013	18/02/2008 – 23/09/2013
Boeing	NYSE	BA	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Caterpillar	NYSE	CAT	06/05/1991 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Chevron	NYSE	CVX	03/01/1989 – 01/11/1999; 19/02/2008 – 29/12/2017	03/01/1989 – 01/11/1999; 19/02/2008 – 29/12/2017	03/01/1989 – 01/11/1999; 18/02/2008 – 29/12/2017
Cisco Systems	NASDAQ	CSCO	08/06/2009 – 29/12/2017	08/06/2009 – 29/12/2017	08/06/2009 – 29/12/2017
Citigroup	NYSE	C	01/11/1999 – 08/06/2009	03/01/1989 – 08/06/2009	03/01/1989 – 08/06/2009
Coca-Cola	NYSE	KO	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
DowDuPont	NYSE	DWDP	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
ExxonMobil	NYSE	XOM	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
General Electric	NYSE	GE	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Goldman Sachs	NYSE	GS	26/09/2013 – 29/12/2017	26/09/2013 – 29/12/2017	23/09/2013 – 29/12/2017

Table 3 continued

Company	Exchange	Symbol	Period in DJIA	Extended period in DJIA	Period in portfolio
Goodyear	NASDAQ	GT	03/01/1989 – 01/11/1999	03/01/1989 – 01/11/1999	03/01/1989 – 01/11/1999
Hewlett-Packard	NYSE	HPQ	17/03/1997 – 26/09/2013	03/01/1989 – 26/09/2013	03/01/1989 – 23/09/2013
The Home Depot	NYSE	HD	01/11/1999 – 29/12/2017	01/11/1999 – 29/12/2017	01/11/1999 – 29/12/2017
Honeywell	NYSE	HON	03/01/1989 – 19/02/2008	03/01/1989 – 19/02/2008	03/01/1989 – 18/02/2008
IBM	NYSE	IBM	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Intel	NASDAQ	INTC	01/11/1999 – 29/12/2017	01/11/1999 – 29/12/2017	01/11/1999 – 29/12/2017
International Paper	NYSE	IP	03/01/1989 – 08/04/2004	03/01/1989 – 08/04/2004	03/01/1989 – 05/04/2004
Johnson & Johnson	NYSE	JNJ	17/03/1997 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
JPMorgan Chase	NYSE	JPM	06/05/1991 – 29/12/2017	06/05/1991 – 29/12/2017	06/05/1991 – 29/12/2017
Mondelez	NASDAQ	MDLZ	22/09/2008 – 24/09/2012	22/09/2008 – 24/09/2012	22/09/2008 – 24/09/2012
McDonald's	NYSE	MCD	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Merck	NYSE	MRK	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Microsoft	NASDAQ	MSFT	01/11/1999 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Navistar	NYSE	NAV	03/01/1989 – 06/05/1991	03/01/1989 – 06/05/1991	03/01/1989 – 06/05/1991
Nike	NYSE	NKE	26/09/2013 – 29/12/2017	26/09/2013 – 29/12/2017	23/09/2013 – 29/12/2017
Pfizer	NYSE	PFE	08/04/2004 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Procter & Gamble	NYSE	PG	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Travelers	NYSE	TRV	08/06/2009 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
United Health Group	NYSE	UNH	24/09/2012 – 29/12/2017	24/09/2012 – 29/12/2017	24/09/2012 – 29/12/2017
United Technologies Corporation	NYSE	UTX	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Verizon	NYSE	VZ	08/04/2004 – 29/12/2017	08/04/2004 – 29/12/2017	05/04/2004 – 29/12/2017
Visa	NYSE	V	26/09/2013 – 29/12/2017	26/09/2013 – 29/12/2017	23/09/2013 – 29/12/2017
Wal-Mart	NYSE	WMT	17/03/1997 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017
Walt Disney	NYSE	DIS	06/05/1991 – 29/12/2017	03/01/1989 – 29/12/2017	03/01/1989 – 29/12/2017

The data of predecessors for some companies in the DJIA are not available due to different reasons: bankruptcy of predecessor, merger or acquisition, the reissue of a predecessor ticker on the stock exchange, as a result, the old history was erased. Therefore, the period of some companies in the DJIA extended in such a way to cover predecessor and own period in the DJIA. The following companies have an extended period in the DJIA: Alltria Group, American International Group, Caterpillar, Citigroup, Hewlett-Packard, Microsoft, Johnson & Johnson, Pfizer, Travelers, Wal-Mart, Walt Disney.

Notice that some companies were replaced by another one:

Honeywell, Alcoa, American International Group, Mondelez, Hewlett-Packard, Chevron, Goodyear, International Paper, Navistar, Altria Group, Bank of America Corporation, AT & T, Citigroup **were replaced by**

Chevron, Goldman Sachs, Mondelez, United Health Group, Visa, The Home Depot, Intel, Verizon, JPMorgan Chase, Bank of America Corporation, Nike, Apple, Cisco Systems, **respectively**.

In addition, let us consider an example how we rebalance portfolio taking into account **the change in the composition of the DJIA**. For example, the stock of International Paper is in the DJIA over the period from 03/01/1989 till 08/04/2004, where the last date is Thursday. International Paper is replaced by Verizon that is in the DJIA over the period from 08/04/2004 till 29/12/2017.

Since this thesis suggests rebalancing the portfolio on the first business day of each week then we should rebalance our portfolio on Monday, 05/04/2004. We have to take into account the announcement of the DJIA composition change in advance, otherwise, delay in the change of the portfolio composition can lead to a huge loss of capital. As a result, the stock of International Paper is in the portfolio over the period from 03/01/1989 till 05/04/2004, and the Verizon stock – over the period from 05/04/2004 till 29/12/2017. Overall, the following dates when the portfolio composition is changed are adjusted to avoid the expected loss caused by either the bankruptcy of the company that left the DJIA or by a huge stock price fall:

1. 08/04/2004 replaced by 05/04/2004.
2. 19/02/2008 replaced by 18/02/2008.
3. 26/09/2013 replaced by 23/09/2013.
4. 19/03/2015 replaced by 16/03/2015.

To conclude, we have the following 9 dates when the portfolio composition is changed: 06/05/1991, 01/11/1999, 05/04/2004, 18/02/2008, 22/09/2008, 08/06/2009, 24/09/2012, 23/09/2013, 16/03/2015.

All statements above are applied only to weekly rebalancing strategy. However, a reader can do the same transformation of dates when the portfolio composition is changed for monthly, quarterly, semiannually, and annual rebalancing strategies.

APPENDIX F

FOUR ECONOMIC SUB-PERIODS

Table 4. The description of the economic sub-periods

Sub-period	Start date	End date	Brief description
1	03/01/1989	04/11/2001	The economic expansion and a brief recession at the end of the period
2	05/11/2001	02/12/2007	The economic expansion that is changed by a steep recession
3	03/12/2007	07/06/2009	The Great Recession
4	08/06/2009	29/12/2017	The economic expansion (recovery of the US economy)

APPENDIX G

STOCKS DESCRIPTION OVER FOUR SUB-PERIODS

Table 5. The statistical description of stocks over the first sub-period

Subperid: 03/01/1989 - 04/11/2001						
Indicator	Number of trading days	Average annual return, %	Standard deviation, %	Skewness	Kurtosis	Sharpe ratio
Min	504	-2.25	21.9	-1.71	1.53	-0.04
Mean	2961	16.5	32.4	0.027	5.60	0.55
Max	3241	41.5	66.0	0.45	32.2	1.10

Notes: The minimum number of trading days when the stock is in the portfolio is 503 (HD, INTC), the maximum – 3240, the average - 2961. The minimum average annual return of the share is -2.25% of NAV stock, the maximum value is 41.5% of MSFT stock, the average value across all stocks – 12.4%. The standard deviations of annual returns are in the range 21.9% – 66.0%. The riskiest stock is INTC (66.0%), but the most riskless one is XOM (21.9%). Skewness values are in the range -1.71 – 0.45, kurtosis values are in the range 1.53 – 32.2. It means that distributions of annual returns of all stocks are non-normal. The Sharpe ratios are in the range -0.04 – 1.10. The NAV stock has the minimum Sharpe ratio, the MSFT stock has the maximum one, the average value is 0.55.

Table 6. The statistical description of stocks over the second sub-period

Sub-period: 05/11/2001 – 02/12/2007						
Indicator	Number of trading days	Average annual return, %	Standard deviation, %	Skewness	Kurtosis	Sharpe ratio
Min	607	-6.27	16.3	-2.39	0.92	-0.25
Mean	1480	8.40	25.7	-0.16	7.68	0.33
Max	1528	24.5	37.1	0.52	41.1	0.82

Notes: The minimum number of trading days when the stock is in the portfolio is 607 (IP), the maximum – 1528, the average - 1480. The minimum average annual return of the share is -6.27% of PFE stock, the maximum value is 24.5% of HPQ stock, the average value across all stocks – 8.40%. The standard deviations of annual returns are in the range 16.3% – 37.1%. The riskiest stock is INTC (37.1%), but the most riskless one is PG (16.3%). Skewness values are in the range -2.39 – 0.52, kurtosis values are in the range 0.92 – 41.1. It means that distributions of annual returns of all stocks are non-normal. The Sharpe ratios are in the range - -0.25 – 0.82. The PFE stock has the minimum Sharpe ratio, the CAT stock has the maximum one, the average value is 0.33.

Table 7. The statistical description of stocks over the third sub-period

Sub-period: 03/12/2007 - 07/06/2009						
Indicator	Number of trading days	Average annual return, %	Standard deviation, %	Skewness	Kurtosis	Sharpe ratio
Min	52	-229	18.6	-2.01	0.50	-1.76
Mean	354	-22.6	55.5	0.41	4.39	-0.38
Max	380	24.7	136	1.03	22.2	0.29

Notes: The minimum number of trading days when the stock is in the portfolio is 52 (MO), the maximum – 380, the average - 354. The minimum average annual return of the share is -229% of AIG stock, the maximum value is 24.7% of JPM stock, the average value across all stocks – -22.6%. The standard deviations of annual returns are in the range 18.6% – 136%. The riskiest stock is C (136.0%), but the most riskless one is MO (18.6%). Skewness values are in the range -2.01 – 1.03, kurtosis values are in the range 0.50 – 22.2. It means that distributions of annual returns of all stocks are non-normal. The Sharpe ratios are in the range - -1.76 – 0.29. The AIG stock has the minimum Sharpe ratio, the WMT stock has the maximum one, the average value is -0.38.

Table 8. The statistical description of stocks over the fourth sub-period

Sub-period: 08/06/2009 - 29/12/2017						
Indicator	Number of trading days	Average annual return, %	Standard deviation, %	Skewness	Kurtosis	Sharpe ratio
Min	706	-7.3	13.7	-0.92	1.64	-0.21
Mean	1849	13.1	21.8	-0.01	5.15	0.64
Max	2158	28.1	43.0	1.16	19.9	1.41

Notes: The minimum number of trading days when the stock is in the portfolio is 706 (AAPL), the maximum – 2158, the average - 1849. The minimum average annual return of the share is -7.3% of HPQ stock, the maximum value is 28.1% of UNH stock, the average value across all stocks – -13.1%. The standard deviations of annual returns are in the range 13.7% – 43.0%. The riskiest stock is BAC (43.0%), but the most riskless one is JNJ (13.7%). Skewness values are in the range -0.92 – 1.16, kurtosis values are in the range 1.64 – 19.9. It means that distributions of annual returns of all stocks are non-normal. The Sharpe ratios are in the range -   -0.21 – 1.41. The HPQ stock has the minimum Sharpe ratio, the UNH stock has the maximum one, the average value is 0.64.

APPENDIX H

THE COMPARISON OF PORTFOLIOS PERFORMANCE

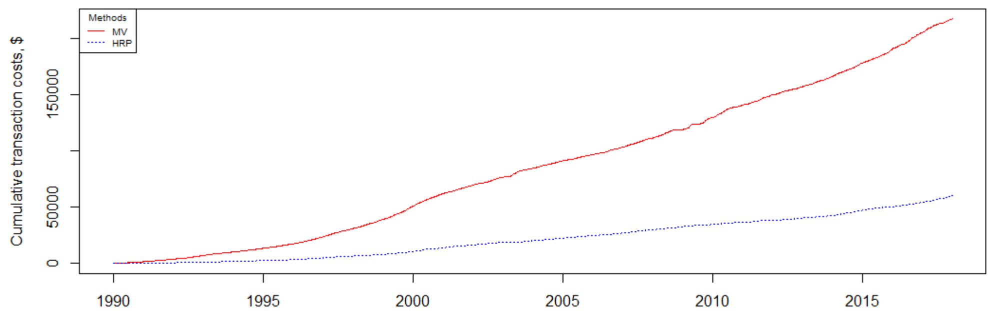

Figure 4. Cumulative transaction costs provided by MV and HRP, $

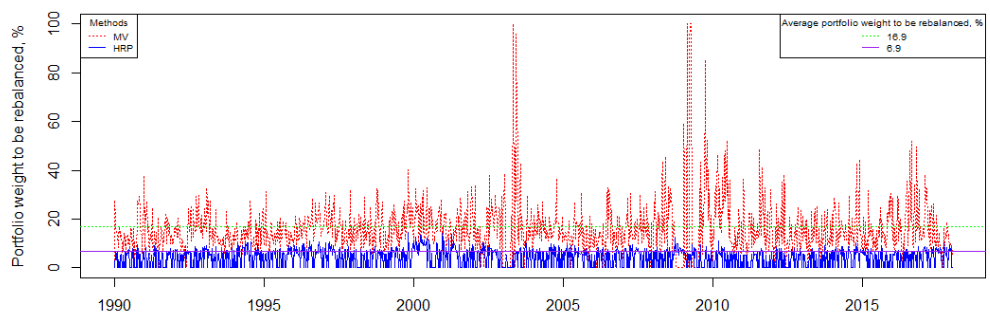

Figure 5. The share of the portfolio to be rebalanced using MV and HRP, %

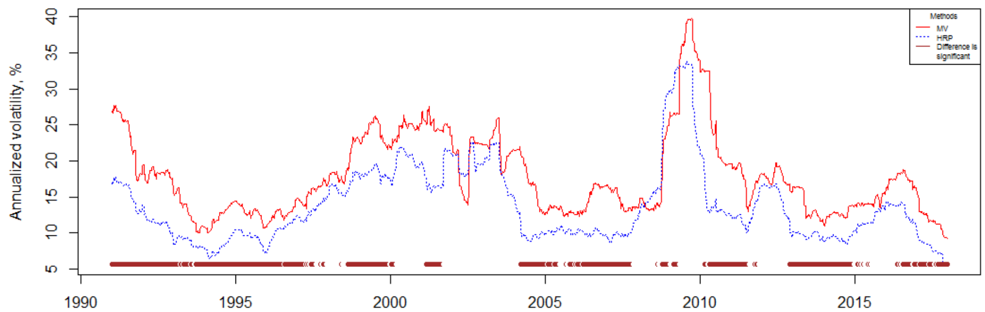

Figure 6. The annualized volatility of annual portfolio returns provided by MV and HRP, %

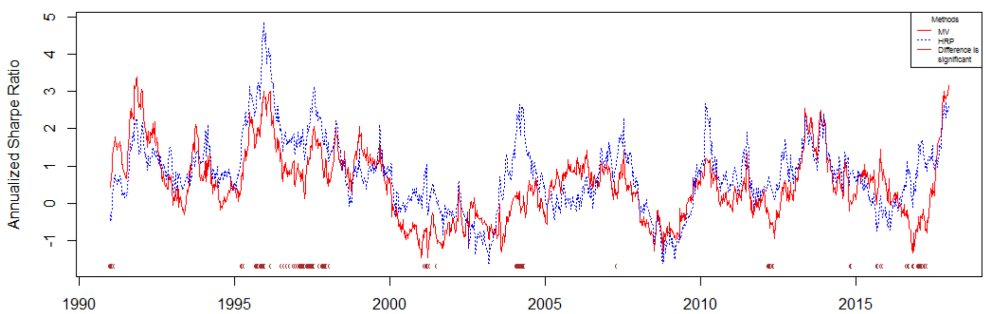

Figure 7. The annualized Sharpe ratio provided by MV and HRP

APPENDIX I

THE SENSITIVITY ANALYSIS OF MV AND HRP

Table 9. The sensitivity analysis of two portfolio diversification approaches over the whole period

Fee	Indicator	No threshold		5% threshold		10% threshold	
		MV	HRP	MV	HRP	MV	HRP
0%	Average annual return , %	11.56	10.43	11.58	10.40	11.65	10.44
	Standard deviation, %	18.73	14.62***	18.72	14.63***	18.73	14.68***
	Sharpe ratio	0.51	0.58	0.51	0.57	0.52	0.57
	CAGR, %	0.19	0.18	0.19	0.18	0.18	0.19
	Cumulative transaction costs, $	0	0	0	0	0	0
	Average portfolio weight to be rebalanced	16	5.3***	16.9	6.9***	19.1	11***
	Rebalancing frequency, %	97.53	100	94.4	60.6	78.6	9.45
0.1%	Average annual return , %	9.86	9.78	9.90	9.87	10.07	10.24
	Standard deviation, %	18.72	14.62***	18.71	14.64***	18.73	14.69***
	Sharpe ratio	0.42	0.53	0.42	0.54	0.43	0.56
	CAGR, %	0.16	0.17	0.16	0.17	0.16	0.18
	Cumulative transaction costs, $	217K	75K***	217K	60K***	208K	16K***
	Average portfolio weight to be rebalanced	16	5.33***	16.9	6.9***	19.1	11***
	Rebalancing frequency, %	97.5	100	94.4	60.6	78.6	9.45
0.3%	Average annual return , %	6.48	8.49	6.53	8.83	6.9	9.88
	Standard deviation, %	18.71	14.63***	18.71	14.64***	18.72	14.69***
	Sharpe ratio	0.24	0.44*	0.24	0.47*	0.26	0.54**
	CAGR, %	0.09	0.14	0.09	0.15	0.099	0.17
	Cumulative transaction costs, $	387K	180K***	387K	150K***	380K	14K***

Table 9 continued

Fee	Indicator	No threshold		5% threshold		10% threshold	
		MV	HRP	MV	HRP	MV	HRP
0.3%	Average portfolio weight to be rebalanced	16	5.3***	16.9	6.9***	19.1	11***
	Rebalancing frequency, %	97.5	100	94.4	60.5	78.6	9.38
0.7%	Average annual return , %	-0.25	5.92	-0.18	6.77	0.60	9.08
	Standard deviation, %	18.72	14.68***	18.72	14.69***	18.74	14.75***
	Sharpe ratio	-0.12	0.27***	-0.12	0.32***	-0.08	0.48***
	CAGR, %	-0.04	0.09	-0.04	0.11	-0.02	0.15
	Cumulative transaction costs, $	388K	278K***	390K	250K***	396K	91K***
	Average portfolio weight to be rebalanced	16	5.3***	16.9	6.9***	19.1	11***
	Rebalancing frequency, %	97.5	100	94.5	60.2	78.6	9.38
1%	Average annual return , %	-5.28	4.01	-5.17	5.21	-4.07	8.49
	Standard deviation, %	18.74	14.73***	18.74	14.75***	18.77	14.82***
	Sharpe ratio	-0.39	0.14***	-0.38	0.22***	-0.32	0.44***
	CAGR, %	-0.14	0.06	-0.13	0.08	-0.11	0.14
	Cumulative transaction costs, $	342K	298K***	343K	281K***	355K	119K***
	Average portfolio weight to be rebalanced	16	5.3***	16.9	6.9***	19.1	10.99***
	Rebalancing frequency, %	97.5	100	94.5	60.1	78.4	9.38

Notes: The difference between a particular indicator of MV and HRP is statistically significant at: * - 90% confidence interval (CI), ** - 95% CI, *** - 99% CI.