OPTION PRICING WITH FRACTALS:

AN EMPIRICAL ANALYSIS

by

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Abstract

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Classical Black – Scholes model, however still widely used in the financial circles, is known for its inconsistencies in modeling actual option markets. Besides assumption of the normal distribution of the underlying assets and perfect liquidity of the market, classical Black-Scholes provides poor modeling of in-the-money and out-of-the money the options, as well as does not account for the term structure of the options. Current work concentrates on the Fractional Black-Scholes option pricing model, that relaxes classical assumptions and provides theoretically valid modeling of volatility smile and term structure of options. Following the first empirical work, concerning application of the model to the actual options data, appeared in Mare et al (2017), current work provides further investigation of the Fractional Black-Scholes model, as well as it's comparison with classical Black - Scholes and Stochastic Alpha Betha Rho models. The estimation of FBS model parameters are closely studied, as well as statistical tests to prove the significance of the model estimated. Model is fitted to the S&P500 index options data and the performance is compared with classical BS model and SABR model in terms of root mean squared error and mean average percentage error.

Results give an evidence that fractional model have better accuracy then classical Black-Scholes model in terms of prediction error, and is comparable

to the SABR model. Among other advantages of FBS modes, is strong heuristical approach to the model parameters, which allows to interpret them in order to understand the market behavior. Solid mathematical background behind FBS model allows researcher to execute statistical testing of the model validity. Closed-form pricing formula of Fractional Black-Sholes model provides simplicity of estimation procedure.

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GLOSSARY

EMH. Effective Market Hypothesis, theory about efficient financial markets.

FMH. Fractal Market Hypothesis, theory explaining inefficiencies observed on the financial markets.

SABR. Stochastic alpha betha rho option pricing model.

BS. Black-Scholes option pricing model.

FBS. Fractional Black-Scholes option model.

MAE. Mean absolute percentage error, used to measure estimation or prediction error.

RMSE. Root mean squared error, used to measure estimation or prediction error.

S&P500. Standart and Poor index representing aggregate index of top 500 companies in US market.

Chapter 1

INTRODUCTION

The first theoretical model of option prices was developed by Black, Merton and Scholes (1973). The Black –Scholes option pricing model is considered to be classical benchmark model for options even in the modern financial world. Though relying on theoretical assumptions of normality, perfect liquidity, market efficiency and fixed volatility, it fails to predict all price fluctuations on the option markets, especially in some edge cases, when either time to maturity or strike price is low (Ray 2012).

There are many different option pricing models, which were developed for the purpose of more precise estimation of options fair price: local volatility models (where price is predicted for every small group of options with specific characteristics) and stochastic volatility models (where volatility of option is assumed to be stochastically formed).

This thesis is concentrated on the new model for option pricing, called Fractional Black-Scholes model, which belongs to a class of stochastic volatility models. The uniqueness of the model is explained by complicated fractional mathematics at the model foundation, complete approach "from scratch", which means that the model functional form is derived from new assumptions on the underlying stochastic process. The privilege of the model is a strong heuristical explanation of the new fractional parameter introduced into the model, which helps to describe the behavior of option market. Also, unlike many other stochastic volatility models, the fractional model provide natural extension to the functional form of the classical Black-Scholes model.

The research question of the current work is whether the generalized form of the classical BS model show better performance in options price prediction in comparison to another modernizations of the BS model.

The research is aimed to explore the validity of new predictive models for determination of options prices, as well as compare model performance with classical and modern option pricing models. The results of this research will be useful for the hedge fund companies and private investors that highly rely on the quality of financial models they make use. The methodology described will be also useful for general modeling of options market structure, which will provide qualitative understanding of the underlying processes driving the options prices.
For comparison with the Fractal Black-Scholes model, classical Black-Scholes as well as Stochastic Alpha, Betha, Rho (SABR) models are used. SABR is also a further modification of the classical Black-Scholes model, with a different form of the underlying stochastic process. Underlying geometric Brownian motion process is modified to have two stochastic state variables (for volatility and price motions), which have a correlation parameter between each other (the concept that is similar to fractional Brownian motion).

Fractional mathematics started to be leveraged for modeling purposes from the 1980. But vast amount of initial researches was devoted to the exploratory analysis of fractional processes, their usability and application to economics and finance, analysis of economic indices and processes that can be modeled using fractals. However, due to complicated mathematical foundation, there were lack of works that were transforming abstract mathematical apparatus to the research instrument. In particular, some of works discovered inconsistency of fractal models for financial modeling purposes (Rogers 1997). After 2000's, updated mathematical apparatus (Rogers et al 2000) allowed to confute previous misleading result. To summarize, almost 30 years after Mandelbrot (1987) explained possible benefits of using fractional models in economics, those models were applied to the options markets (Maré et al 2017).

The fractional models, previously used for the exploratory purposes and time series forecasting models, proved to add more understanding of the nature of economic processes. The fractal approach is closely connected with the modern field of Economics – Behavioral Economics. The Fractal Market Hypothesis (Mandelbrot 1987) provided explanation of various market inefficiencies, providing the statement, that only balance of long-term and short-term players on the market can insure the market efficiency. Possible deviation from the balance is displayed in the so-called Hurst exponent – or fractal parameter of the economic process.

Option pricing model, developed under assumption of the fractal nature of the financial process, allows to model inefficient markets, where distribution of market returns is not normal.

The contribution of the thesis is application of Fractal Black-Scholes model to classical European option markets (S&P500 options), comparison of emerging fractal modeling to another classical and modern option pricing approaches. Results show, that fractal Black-Scholes model have accuracy, comparable to another modern option pricing models, but provide more interpretability.

The work is structured in the following way: Chapter 2 presents the literature review on the topic option pricing models and fractional modeling; in Chapter 3 there is a methodology applied in current research, Chapter 4 contains the information about the data used. The estimation results are presented in Chapter 5, while Chapter 6 presents the discussion of obtained results and possible policy implications.

Chapter 2

LITERATURE REVIEW

Right up to the late 80s, financial modeling and prediction techniques, based on the mathematical models and programmed trading agents, became increasingly popular due to the rapid development of computational facilities, as well as the increased rate and volume of global financial trade.

Following the classical economic theories, in particular the Efficient Market Hypothesis (EMH), Black and Scholes (1973) developed a model to predict prices of options. This model is still used in modern financial modeling of options and is considered as a classical model for options prices determination.

2.1. Drawbacks of the Black-Scholes model

As generalized by Ray (2012) the Black-Scholes model was based on some fundamental assumptions and restrictions put on the option markets. Among them are:

- Efficient markets: any inefficiency in the market will be transferred to the profit of some trading agent, in such a way bringing market back to the efficient state.
- Perfect liquidity: there is no considerable lags or obstacles for agents on the markets to implement their decisions. Perfect liquidity means that it is possible to buy or sell any amount of option or its fractions at any given time.

- Rational agents: agents on the financial market are assumed to be rational profit-maximizers, who act in the best possible way to achieve their goal.
- Normal distribution: prices of stocks are distributed according to the Gaussian distribution, which is considered as a basic distribution assumed by most models.
- Constant volatility: the volatility of the underlying asset follows the geometrical Brownian motion, with classical (since Newton times) possible growth of volatility as square root of time.
- Market homogeneity: perfectly efficient markets consist of agents with the same objective goal (maximize profits), same level of rationality, and as a result, same behavior.

Significant failures of the classical model were discovered: failure to explain most of price fluctuations (Teneng 2011), especially those at close to maturity and low strike prices (Hagan 2002), unrealism of constant volatility assumed and therefore no volatility smile characteristic, simplified assumption of underlying Markov process (Hakan 2005). Among the most crucial assumptions of the model was perfect liquidity and efficiency of the markets. Therefore, modeling approaches shifted from the classical view on financial modeling, to new methodologies and techniques.

2.2. Alternative modeling approaches and SABR model

In particular, with new possibilities for using heavily computational models, Monte-Carlo option pricing simulation models (Mills et al 1992) and binomial approximation model (Cox et al 1976) emerged, which allowed to account for stochastic volatility.

Work by Bakshi et al (2003) examined an empirical performance, comparing the classical model in comparison to more recent local volatility models (where volatility is assumed to be locally constant) and stochastic volatility models (where volatility is modeled as a stochastic process).

Considerable drawbacks of local volatility models were claimed by Hagan et al (2002), especially in illiquid markets: "dynamic behavior of smiles and skews predicted by local volatility models is exactly opposite to the behavior observed in the marketplace", which means that models do not correctly display a dependence between implied volatility and strike price (which resembles smile form when graphed). The nature of the problem, according to authors, is the underlying assumption of Markow process as the basis for the price evolution. Inappropriate behavior of the models is evidence, that Markovian processes can not be used for modeling of some options. According to the authors, there are three ways to proceed – either assume non Markov stochastic process as the basis for price change identification, or try to keep it Markovian, but use some different process instead of Brownian. And the third approach is to develop a new model based on two – factor approach. The last one was chosen for the SABR model – instead of one equation connecting price with volatility and time, authors proceeded with the system of two equations, the first one described the behavior of price as the Brownian motion, while the second described volatility as an independent Brownian process, but correlated with the first one. Following new initial assumptions, a new model describing the implied volatility was derived. The model called Stochastic Alpha Beta Rho option pricing model, was chosen by the author of the thesis to be a comparison model in the family of modern approaches. SABR model introduces a functional form alternative from other modern models (for example, Heston model (1993)), which provides some more beneficial generalizations like a lognormal distribution, no mean reversion and scale variance (Alexander and Nogueira 2007).

Hagan (2002) stated the importance of the SABR approach for modeling of the options. Unlike another models (binomial model of Cox, Ross and Rubinstein (1976), Heston (1993) stochastic volatility model), which provide the alternative approach and as a result, somewhat different functional form of relationship between exogenous parameters and option price, the model provides a substantial improvement to the classical BS model. The aim of SABR model is to an provide implied volatility parameter, which replaces a corresponding parameter in the original BS formula. A simple procedure of fetching the model parameters consists of three procedures: estimation of the exponent for the forward rate, obtained by simple OLS; estimation of the initial variance, volatility of variance and correlation between two Brownian processes assumed in the model, one for forward rate, the other one – for the variance of volatility. Using fetched parameters, volatility smile and skew are obtained to illustrate behavior of the process modeled. Another important characteristics of the SABR model are Greeks, which characterizes the modeled option. Parameter Vega describes the sensitivity of the option price to volatility. Parameter Delta explains how the option price reacts to the strike price.

West(2004) provided the technical details about the applied usage of SABR model, numerical algorithms needed for the parameters estimation are explained – log-log estimation procedure for estimating the price evolution parameter, Tartaglia method for solving a third-order equation to obtain implied volatility from at-the-money volatility, Nelder-Mead simplex search algorithm and Newton – Rhapson solver for estimating the last two parameters - correlation between stochastic processes and volatility variance. The author explains that SABR model, though constructed for American options, can be greatly used for European options studied in current thesis, because it is suboptimal to withdraw the money before the end of traded period.

2.3. Fractional approach to modeling and Fractional Black-Scholes model

As emphasized by Mandelbrot (1987), wrong assumptions about constant volatility and underlying processes in classical models have more conceptual than technical background. And one crucial assumption is that financial markets are efficient.

The theory stated the evidence about faults of the classical financial theory, which undervalued the effect of rare events. According to the classical view that tried to fit the distribution of stock prices to the normal "bell-shaped" curve (Gaussian curve), there are rare events placed on the tails of distribution and are not accounted for, because no financial model or stock portfolio can elucidate for 100% of all risks or finance shocks. Mandelbrot discovered that actually, distributions of many stock prices do not fit well the normal distribution. Moreover, a lot of significant and astonishing changes on markets have considerable magnitude and low possibility to occur. As a result, most of financial crashes happened because of events placed at the tails of distribution. The idea of fat tails, an important influence of the rare events on the economy, repeatable occurrences of the rare events laid the foundation of new theory, which was called the "Fractal Market Hypothesis". This theory relaxed some assumptions of the classical theory:

- Non-efficient markets: there is a certain possibility for the financial market to become inefficient, for example, in the case of overheated expectations by agents, interacting on the market. There is not enough information symmetry, described by the EMH, and moreover, not all public information is accessible by financial agents, as well as private information can be revealed by accident. As a result, there is some possibility for non-realized profit opportunities that keep market working inefficiently.
- Illiquidity: markets cannot be perfectly liquid, because of political, financial, physical constraints, design of stocks (for example it is rarely possible to sell any fraction of asset). Because perfect liquidity assumes

perfectly competitive markets, and the last phenomena cannot be observed in reality, perfect liquidity cannot be observed as well.

- Irrational agents: because the agents interacting on the markets are still mostly humans, they are subject to panic, emotions and sometimes irrational behavior. Their decisions are influenced by subjective goals and utilities, which do not always result in rational decisions from the point of view of the economy.
- Non normal distribution: a lot of price distributions are better explained by generalized version of normal distribution: distribution of Pareto-Levi. Gaussian classical distribution is just a particular example of a more general law, which can be perfectly fitted to the skewed distributions with long tails.
- Non-constant volatility: the assumption of constant volatility cannot be valid, especially in the long run. Volatility can change because of irrational behavior on the markets, changed the market structure or some unpredicted events.
- Heterogeneity of the markets: from the point of view of the fractional theory, financial markets can be stable and efficient, if there are different types of agents present on the market: short-term investors and long-term investors. As the decisions made by economic agents are highly dependent on their financial horizon, markets are stabilized, if actions of short-horizon agents are compensated by long-term looking agents. If some market is overcrowded with short-looking ones, the market is highly volatile and trend-following, which may result into deep deviations from the efficient price and form so-called "bubbles"; if there is only long-term agents on the market, it is highly illiquid and non-flexible. Thus, only a perfect mix of agents of different trading horizons can create perfectly efficient, liquid markets.

Mandelbrot (1987) compared markets to turbulent seas, implying changing in time volatility, non steady movement of prices, market inefficiencies, scaling effect of time. The comparisons with the nature provides introduction to the concept of fractals, and the idea, that financial markets are similar to the behavior of various natural phenomena in the world. The history of shifts from classical to modern views on the market modeling were outlined, visualizing some misconduct of classical approach to market modeling, as well as providing examples of utilizing the fractional approach. The book served as a main motivational cornerstone, which inspired the author of the thesis to conduct a further research in this field.

Mandelbrot's follower E. Peters (1994) in his book gave more scientific examples on the role of fractals in the analysis of financial markets. A strong fundamental basis was explained, starting from definition of fractals and fractional dimension, proceeding with understanding the nature of fractional time series and inconsistency of normal distribution in application to financial indices (Dow-Jones, S&P 500). R/S model and V-statistics were introduced to help qualitatively and quantitatively analyze time series in order to identify the fractional parameter of processes and specific properties, such as fractional cycles.

The findings of Mandelbrot and his followers faced a strong critique and were undervalued over a long period of time. However, researches were aimed to explain the reasons for financial cataclysms, discovered certain evidence of the theory. For example, the research done on the movements of stock prices, discovered the presence of so-called "excess volatility" (Shiller (1981)), the statement that movements of stocks have larger amplitude than underlying assets, which gives the evidence about irrational influence of human behavior on the movements of financial markets. There were a lot of other researchers stating the problem of the validity of the classical approach to understanding the financial system : Peters(1994), Kahneman and Tversky(1974), Hommes (2006), Taylor(1990), Frankel(1986) and others.

Longarela et al. (2007) provided the evidence about cases of non-efficiencies in option prices. The authors provided the evidence, why non zero bid-ask spreads on option market arise and explained them by so-called "quote inefficiencies". They developed theoretical explanation and calculation methods to evaluate the so-called "market discrepancies", that explain inefficiencies in the pricings of put and call options. The authors state that puzzling systematically inefficient behavior in the options markets occurs, while examining the data about European options on the Dow-Jones EURO STOXX 50 index, DAX index, options on the E-mini S&P500 futures.

Jarrow et al (2007) provided strict mathematical proofs for the possibility of existence of market inefficiencies, in particular bubbles for call options, explaining that there can exist call option bubbles (no put option bubbles are possible) and their magnitude is equal to the magnitude of the underlying asset bubble . The significance of the paper is a suggestion for utilizing data about options call quotes, because there is a mathematically proven possibility to find huge inefficiencies.

A particular emphasis of the current study is laid on the fractional approach to the modeling. The idea of "self-similarity", proposed by Mandelbrot (1987), particularly well fits to the analysis of financial time series. Indeed, as financial time series are very volatile, there is an observed similar behavior in the financial series on different scale levels. The approach proposed by Mandelbrot, and developed by his follower, E. Peters, gave rise to the methodology based on heuristics. According to this, the geometric Brownian motion, as a stochastic process to model stock movements, is replaced by the fractional Brownian motion with a special parameter H (Hurst parameter, possessing the name of the English researcher Harold Hurst, who discovered the parameter).

The parameter can have a value in the unit interval. If H is less than one half, the financial process is assumed to have a short time memory, with strong fluctuations and interchanging directions of movement. The current increase of the process may result in the further decrease of the process. If H is greater than one half, the process is characterized by a long memory, the current increase in the process will correspond to its future increase (Sviridov et al 2016). H parameter, which exactly equal one half, corresponds to the completely random process, characterized by the geometric Brownian motion.

We can apply the corresponding heuristics to the structure of the financial markets. Indeed, we can connect a short time memory to the prevailing amount of agents with a short trade-horizon. The long memory process will correspond to the prevailing amount of long trade horizon players on the market.

This heuristics lay at the core of the fractional Black-Scholes model for the options price prediction. The modified stochastic process – the fractional Brownian motion instead of the geometric Brownian motion, resulted in the generalized form of Black-Sholes equation with the strong heuristical approach to the analysis of the financial process.

The fundamental obstacle for the application of the fractional Brownian motion, laid in the fact, that as a fBm can have an infinite variation (in more scientific notation – it is not a semi-martingale). This fact leads to a problem that classical stochastic calculus could not be applied in the case of the fractional Brownian motion.

After Lin (1995) showed that it was possible to do an approximation of fBm using the "fractional Gaussian noises" approach, Rogers (1997) provided a proof that financial models based on fBm were ambiguous and inappropriate for modeling of prices evolution, because they would have arbitrage opportunities and, therefore, were not valid for the market modeling (modeled market would be always incomplete and, therefore, would not fit to the real financial markets). After such results, fractional models were claimed to be inappropriate for the financial modeling.

The attitude to the fBm – based modeling changed after introducing a new type of stochastic integral for fractional processes (Dunkan et al 2000) . Hu and Oksendal (2000) showed that the fractional Black-Sholes model derived by means of using an appropriate integral has no arbitrage and therefore is an appropriate model. Hu and Oksendal derived the formula of the European options for the initial time of option placement.

Necula (2002) extended the formula to price options for any time period, starting from zero till the maturity date. The formula is valid for any Hurst parameter more or equal to one half (for the long memory process). Necula showed that option price depends not only by time to maturity, but memory characteristic of the underlying asset pricing process.

Using fractional Brownian motion and Wick-Ito-Skorohod integral, concise deduction of the final fractional Black-Scholes formula for the optimal call price of the option is provided in the book by Mishura(2008).

Another theoretical problem is related to the estimation of Hurst parameter. Taqqu et al (1995) provided a comparison of different approaches to calculation of Hurst parameter (R/S analysis (Mandelbrot and Wallis 1969), periodogram method, Higuchi (1988) method).

Li and Chen (2014) proposed two alternative approaches: the use of the inverse fractional Black-Scholes formula to derive the implied Hurst parameter, and another approach, which does not depend on any model chosen. Li and Chen made an emphasis on the value of the model – so called independent approach, which allows to capture a long time memory effect more precisely. They also underlined the necessity for the appropriate statistical testing of the validity of the Hurst parameter obtained.

Regarding empirical results on application of the fractional Black-Scholes formula, there are several works. Flint and Maré (2017) verified the performance of FBS on the South-African Options market. Among other results, there is a particularly interesting conclusion that the Hurst parameter is uncorrelated with implied volatility, thus the Hurst parameter captures a long-memory component independently.

The Fractional Black-Scholes model is an alternative way to enhance functionality of the classical BS model. The fundamental difference between the SABR model is that in the FBS the initial stochastic relation is not modified. Instead, non-Markov processes are allowed to act as stochastic generators. Using a new fractional parameter, the underlying process is allowed to have any correlation, starting from Markov one step – to infinity.

Chapter 3

METHODOLOGY

Overall, the price of an option is dependent on six observables: price of the underlying stock (spot price), strike price of an option, number of days until option expiration (maturity date), interest rate on risk free asset, dividend yield of the stock and volatility of the underlying asset, as unobservable factor.

3.1. General methodology

Spot price of the asset, strike price of the asset, maturity date, interest rate, dividend yield are the given parameters, characteristics of the option, which are directly observed on the market. In comparison volatility is the realization of the random process (volatility changes in time), that needs to be modeled.

In order to approximate risk free interest rate, government bond yields are usually taken, with appropriate time horizon. It means that maturity period of the government bond must correspond to the maturity period of an option.

Dividend yield of the stock is calculated according to financial statements of the company that issues the stock, and for the case of index-based stock, it is calculated as the average yield of index. Data on monthly dividend yield of stock can be easily obtained online for the most of stocks. For current use data is obtained from the online source (official S&P500 data).

There are three models, which are subject to detailed study in this thesis: classical Black-Scholes model, fractional Black-Scholes model and SABR (stochastic alpha, betha, rho) model. The goal is to compare the model in terms of the empirical performance (modeled option prices versus actual prices on options).

In order to obtain option prices estimation, one needs to fit model parameters. There are parameters, specific to every model. Let's define a vector of parameters that need to be fetched as θ, and vector of known parameters M.

Then, the estimated option price can be described by generalized formula:

$$P = F(\theta, M) \tag{1}$$

(vector M includes given standard price parameters in the formula, such as price of the underlying asset, strike price, time to maturity etc.) The vector of parameters θ is specific to every model studied

Classical Black-Scholes model :

$$\theta = (\sigma), \tag{2}$$

where σ is volatility parameter

The only parameter that needs to be estimated for the classical BS model is the volatility. As one of the assumptions of the classical model is the constant volatility of the underlying asset, one needs to adjust the parameter for every time period in order to obtain a precise estimate of the option price.

Fractional Black-Scholes model:

$$\theta = (\sigma, H) \tag{3}$$

In the Fractional Black-Scholes model, volatility of the underlying asset returns is not constant, but it is dependent onto fractional parameter H – called memory of the process, and determines the behavior of the price. If H

< 0.5, price movements are named antipersistent, they change direction of movement frequently. If $H > 0.5$, process is called persistent. It remains its behavior (upward or downward slope), during long time. If H=0.5, process is a completely random – white noise.

SABR model:

$$\theta = (\sigma(\alpha, \beta, \rho, \upsilon)) \tag{4}$$

In the SABR model, volatility of the price movements depends on the four parameters. Alpha measures volatility of the price movements, betha is a skew, rho is an correlation between two Wiener processes, one for price movement, second for the volatility movement. Nu measures so-called "stochasticity of variance".

So, SABR model, as well as Fractional Black-Scholes model put more emphasis on the modeling of the volatility. However, Fractional Black-Scholes have more intuitive construction framework, which gives the researcher ability for the deep understanding of the model, and as the result, implicit control over the model parameters.

To compare the models, we need to use same methodology for estimating the model parameters, at least for the estimation of the volatility parameter, which is needed for all three models.

Volatility is considered to be one of the major non-observables for the option price. As the volatility modeling in the SABR model is based on the calculated implied volatility, using the inverted Black-Scholes formula, we will use the same methodology for all three models.

Firstly, we calculate implied volatility, using so called inverse Black Scholes formula (BS is inverted numerically). In other words, one fits the volatility parameter in the BS formula to the extent, when fitted option price will be close to the actual option price.

Secondly, estimation of the volatility parameter is based onto the minimization problem:

$$\min_{\sigma} \sum_{i=1}^{N} (\sigma - \sigma_{BSi}^{IV})^2, \tag{5}$$

where σ is model volatility,parameter, σ_{BS}^{IV} is implied volatility calculated using inverted BS formula.

Following from previous equation, best estimator for the volatility parameter is

$$\sigma = \frac{1}{N} \sum_{i=1}^{N} \sigma_{BSi}^{IV} \tag{6}$$

Calculated volatility parameters on the daily basis, are utilized in all three models in order to use common estimation framework in BS, FBS and SABR model.

As accuracy of any financial model is the metric that shows how the modeled prices coincides with the actual prices, we will compare predicted prices on options with the actual prices, as observed in the data.

Accuracy of the model is determined by both model fit (in-sample prediction) and prediction accuracy (out-of sample prediction). In order to estimate accuracy of the model, we will use several metrics (RMSE, MAE), using the following error formulas:

RMSE (root mean squared error):

$$RMSE = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (P_i^{predicted} - P_i^{real})^2} \tag{7}$$

MAPE (mean absolute error):

$$MAPE = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{P_i^{predicted} - P_i^{real}}{P_i^{real}}\right| \quad (8)$$

3.2. Models discussion

Classical Black-Scholes model

Classical Black-Scholes model assumes that there is exactly one riskless asset and option on the risky asset on the market. Using the hedge strategy, a player can diversify his savings using both assets.

Derivation of an option price model includes the assumption, that the price of the asset under option have the data generating process, described by the next formula:

$$dS = \mu S dt + \sigma S dz, \quad (9)$$

where mu is a price drift, z is geometric Brownian motion, with mean 0 and variance σ, S is a stock price.

Final formula for the call option price under Black-Sholes model can be seen from the next formula:

$$C(f, K, \sigma, T) = [SN(d_1) - Ke^{-r(T-t)}N(d_2)] \quad (10)$$

$$d_{1,2} = \frac{\ln S/K + (r-\delta)(T-t) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}, \quad (11)$$

where S is a current price of the underlying asset, Φ is a Gaussian distribution function, K is a strike price ,T is maturity date , t is a current date, r is an interest rate, δ is dividend yield, σ is an option volatility

The formula for the put option price can be derived from put-call parity and equals to

$$P_B(f, K, \sigma_B, T) = e^{-rT}[-SN(-d_1) + KN(-d_2)] \qquad (12)$$

Overall, the price of an option is dependent on six parameters: price of the underlying stock (spot price), strike price of an option, number of days until option expiration (maturity date), interest rate on risk free asset, dividend yield of the stock, volatility.

In order to approximate risk free interest rate, government bond yields are usually taken, with appropriate time horizon. It means that maturity period of the government bond must correspond to the maturity period of an option.

Dividend yield of the stock is calculated according to financial statements of the company that issues the stock.

Volatility parameter is common for all three models and is calculated according to the methodology described at the beginning of the chapter.

Fractional Black-Scholes model

The Fractional Black-Scholes model, used for estimating the option price, states that several parameters influence the price of an option: the price of the underlying asset and its volatility, strike price of option, time to maturity, interest rate of risk-free asset (for example, government bonds),the fractional parameter of the financial market, which is called Hurst parameter.

The functional form of the model can be seen in the next formula:

$$C(t, S) = \Phi\left(\frac{ln\frac{S}{K} + (r - \delta)(T - t) + (T^{2H} - t^{2H})\frac{\sigma^2}{2}}{\sigma\sqrt{T^{2H} - t^{2H}}}\right) -$$

$$-e^{-r(T-t)}\Phi\left(\frac{ln\frac{S}{K}+(r-\delta)(T-t)+(T^{2H}-t^{2H})\frac{\sigma^2}{2}}{\sigma\sqrt{T^{2H}-t^{2H}}}\right), \quad (13)$$

Where S is current price of the underlying asset, Φ is a Gaussian distribution function, K is a strike price, T is a maturity date (in years from the beginning),t is a current date (in years fraction), sigma is a standard deviation of the underlying price, H is a fractional Hurst parameter, H ϵ [0,1], r is an interest rate, δ is dividend yield.

All parameters, except the Hurst parameter and volatility are exogenous parameters, which should be present in the dataset for estimation. However, problems with estimation of two parameters may well arise. As these parameters are subject to the researcher's effort to be estimated, they are called implied parameters.

Standard deviation measures the magnitude of uncertainty about the future price of option. The longer is the period to option maturity and higher deviation of prices of the underlying asset, the higher is the price of the call option, as there is more probability that the price of the underlying asset will rise. If time to maturity is short and deviation is small, the low price on option resembles the fact that it is highly unrealistic for the price of the underlying asset to change dramatically.

As the classical Black-Scholes model tends to underestimate options close to maturity and overestimate options with a high strike price, the implied volatility is estimated to achieve higher accuracy of the model.

To overcome the limitation of the classical Black-Scholes model, in particular, the assumption about constant volatility, the practitioners approach is to estimate locally constant volatility by using sliding window.

The process followed in the current study is as follows – take the time frame (for example 30 days) of the underlying asset price, calculate volatility, and assume that volatility to be the same for 31st day. Then move the time window one step ahead and repeat the process again.

A comparative method is to estimate the implied volatility parameter (simply the volatility, for which the classical Black –Scholes model will produce the same option price as observed in the market), and then calculate the regression:

$$\ln[\sigma_{BS}(\tau)] = \ln(\sigma_f) + \left(H - \frac{1}{2}\right)\ln(\tau), \tag{14}$$

Where t is a time to maturity, σ_f in the left hand side stands for implied volatility and H is a Hurst parameter (Li and Chen 2014). By using the estimated regression it is easy to extrapolate the value of volatility to the future unknown period for the price prediction. Both methods are used in my work.

However, the Hurst parameter estimated simply with the regression can be subject to the bias due to non stationary data. Moreover, some additional statistical verification is needed (except of F - statistics), to prove that the underlying asset pricing process can be modeled with the fractional Brownian motion.

So, to obtain the Hurst parameter, some preliminary steps should be taken. Firstly, logarithmic difference is needed to be taken in order to detrend the data and obtain the Brownian process, which is assumed as the underlying data generation process for the price of the underlying asset.

$$z(t) = ln\left(\frac{x_t}{x_{t-1}}\right), \tag{15}$$

Where x is the underlying asset price.

The next step is to obtain the initial estimate of the Hurst parameter, using methodology of R/S (rescaled range) analysis (Taqqu et al 1995) using the formula

$$\frac{R}{S}(n) := \frac{1}{S(n)}\left[\max_{0\le t\le n}\left(z(t) - \frac{t}{n}z(n)\right) - \min_{0\le t\le n}\left(z(t) - \frac{t}{n}z(n)\right)\right], \quad (16)$$

where n is a length of sliding window chosen, z is a logarithmic difference of asset prices, S is a standard deviation of prices. According to the methodology, rescaled range should be calculated for each subsequent period and using the simple OLS, Hurst parameter should be calculated by using the formula:

$$\mathrm{Log}(R/S)_n = a + H * \log(n), \quad (17)$$

where a is the slope of regression, H is an estimate for Hurst parameter

The package "fractal" in R can be used for calculation. However, the obtained estimate is only initial, and should be used as an initial parameter in the next methodology, which obtains more precise value, but needs the initial guess for computation.

Next step is further calibration of the Hurst parameter obtained, using methodology developed by Nourdin et al (2010).

Final step is verification of the statistical significance of the H parameter obtained. Hypothesis, that underlying asset prices can be modeled with fractional Brownian motion, with Hurst parameter obtained. Details of the estimation procedure are provided in Appendix A.

After estimation of the parameter H, the Fractional Black-Sholes model can be applied to arrive at the option prices.

SABR model

According to the Hagan et al (2004), SABR model is based on the different generation process and assumed to consist of two processes, which have their own stochastic Markov Brownian motions.

Option price is given by the standard BS formula:

$$C(f, K, \sigma, T) = [SN(d_1) - Ke^{-r(T-t)}N(d_2)] \quad (18)$$

$$d_{1,2} = \frac{\ln S/K + (r-\delta)(T-t) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}, \quad (19)$$

where K is a strike price, r is risk free rate, δ is a dividend yield, T is time to maturity, S is underlying asset price, σ is a volatility of the underlying asset.

However, σ parameter in the previous formula is replaced by the calculated implied volatility:

$$\sigma_B(K, f) = \frac{\propto\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho^{\beta}v\alpha}{(FK)^{(1-\beta)/2}}+\frac{2-3p^2}{24}v^2\right]T\right\}}{(fK)^{(1-\beta)/2}\left[1\frac{(1-\beta)^2}{24}ln^2\frac{f}{K}+\frac{(1-\beta)^4}{1920}ln^4\frac{f}{K}\right]} \times \frac{z}{X^z} \quad (20)$$

$$z = \frac{v}{\propto}(fK)^{(1-\beta)/2} ln\frac{f}{K} \quad (21)$$

$$X(z) = ln\left[\frac{\sqrt{1-2pz+z^2}+z-p}{1-p}\right], \quad (22)$$

where a, b r, v are the SABR model parameters which are subject to calibration.

There are two steps for calibration of the parameters of SABR model:

Betha parameter can be obtained from the log of previous equation, using OLS:

$$ln\sigma_{ATM} = ln\alpha \; - \; (1-\beta)lnf \qquad (23)$$

The second step is to obtain another three parameters solving minimization problem by using numerical methods (for example Newton-Rhapson method)

$$(\hat{\propto}, \hat{\rho}, \hat{\upsilon}) = arg\ min \sum_i \sigma_i^{mkt} - \sigma_B(S_i, K_i, \alpha, \rho, \upsilon) \qquad (24)$$

After estimating all coefficients, the implied volatility can be calculated. The implied volatility parameter is then placed into the original Black-Scholes formula to obtain a price for the option.

Chapter 4

DATA DESCRIPTION

The dataset taken for investigation contains hourly SPX option prices quotes on S&P 500 in the period of 2005-2006 years. Options in the current dataset have different maturity periods, starting from 3 weeks up to 2 years. There are different strike prices as well, so the current data provide a great opportunity for comparison of different models on different maturities of options.

The data about the risk free rate needed for modeling are taken from the U.S. Government Treasury Bills 30-day rate data for the same period, and extrapolated to cover all intermediate periods. The description of the major variables can be found in the Table 1.

Table 1. Descriptive statistics[1]

Variable	N	Mean	Std	Min	Max
Strike price	66328	1079.99	288.09	70.00	1650.00
Option price	66328	150.93	173.88	0.00	998.50
Underlying price	66328	1168.86	220.12	0.00	1272.74
Standard deviation	66328	240.30	83.24	10.91	379.75
Risk free rate	66328	3.15	0.45	2.42	4.01

4.1. Data preparation

Data was subject to filtering.

Time to maturity. Options with the time to maturity less then 7 and more then 850 days were filtered out from the dataset. Resulting distribution can be seen on the Figure 1.

[1] Source: own compilation using data for S&P 500 options from CBOE

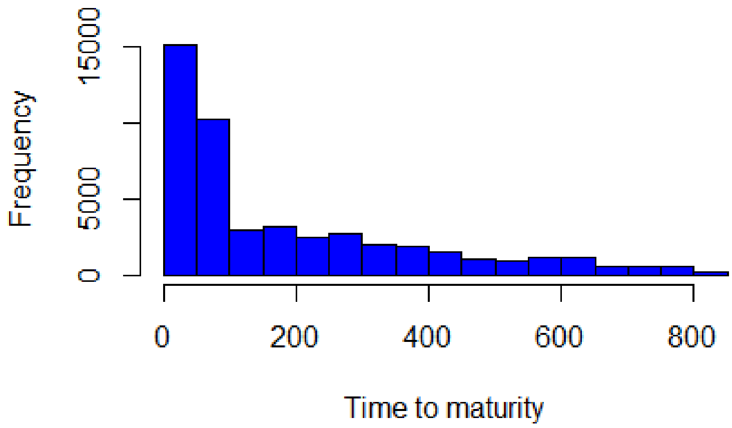

Figure 1. Distribution of the time to maturity of options

Strike price. We filter out the data with both extremely low (less than 500) and extremely high (more then 1600) strike prices, which seem to be an outliers or due to improper data formation. Resulting distribution can be seen on the Figure 2.

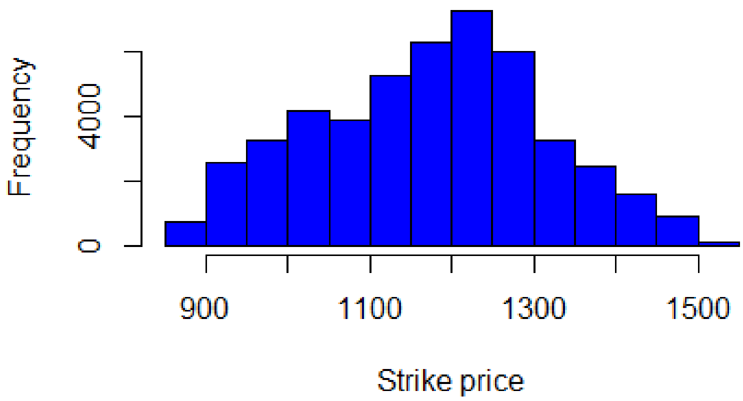

Figure 2. Distribution of strike prices of options

The strike price graph resembles a lognormal distribution, a bit skewed to the right. Distribution of strike prices resembles a belief about the value of underlying index value – indeed, from the Table 1, we can see that the average price of index of 1168.86 dollars.

Moneyness. Moneyness is an ratio of strike price of an option to the current price of the underlying asset. If moneyness is close to one, options are called "at the money". For moneyness less then one, options are called "in the money",if it exceeds one, options are said to be "out of the money". We filer out data based on moneyness parameter due to lack of enough data for the moneyness less than 0.4 and moneyness parameter more than 1.4 (Figure 3).

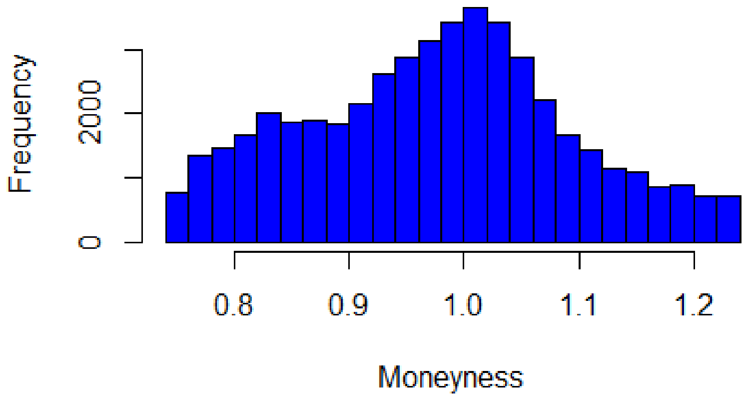

Figure 3. Moneyness of the options

4.2. Resulting dataset description

Resulting dataset is described in the Table 2 (number of data points)

Table 2. Number of data points collected

money-ness / maturity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9 - 1	1-1.05	1.05-1.1	1.1-1.5	Totals
1-40	591	800	1,377	2,353	2,550	3,393	3,460	1,813	1,123	**17,460**
41-90	539	906	1,447	1,871	1,410	2,323	2,868	1,281	1,179	**13,824**
101-250	1,275	1,094	1,579	2,260	1,117	1,145	1,140	916	2,370	**12,896**
251-500	1,243	1,081	1,405	2,231	1,027	1,086	1,067	934	2,352	**12,426**
>500	387	747	757	977	522	698	667	551	1,955	**7,261**
Totals	**4,03**	**4,62**	**6,56**	**9,69**	**6,62**	**8,64**	**9,20**	**5,49**	**8,97**	**63,86**

Distribution of pricing of the options is described in the Table 3 (average prices in groups are given).

Table 3. Average prices by group

moneyness / maturity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9-1	1-1.05	1.05-1.1	1.1-1.5	Totals
1-40	583	404	293	173	91	39	7	1	1	**107**
41-90	587	415	301	183	97	45	13	2	0	**127**
101-250	598	423	306	197	114	68	31	12	1	**187**
251-500	590	431	317	218	142	100	64	35	9	**202**
>500	548	449	345	246	179	139	105	73	26	**193**
Totals	**587**	**424**	**309**	**198**	**111**	**60**	**26**	**16**	**9**	**156**

There are one, two and three year options present in the dataset, however, there is some variation in the expiration dates.

From the graph of historical S&P500 prices, we can see, that 2004 -2005 years used in current research display comparable dynamics to the other time periods, despite structural break in year 2008.

Figure 4. S&P 500 historical prices[2]

[2] source: http://markets.businessinsider.com/index/historical-prices/s&p_500

Chapter 5

ESTIMATION RESULTS

The results from the estimation of classical Black-Scholes and fractional Black-Scholes models can be shown with the next figures. To see the mispricing issues of models, we construct the graphs, where actual prices are sorted in ascending order and corresponding prices obtained from the models are graphed, with price on option as y-axis, and the number of observations in the sorted table as x- axis (Figure 5) .

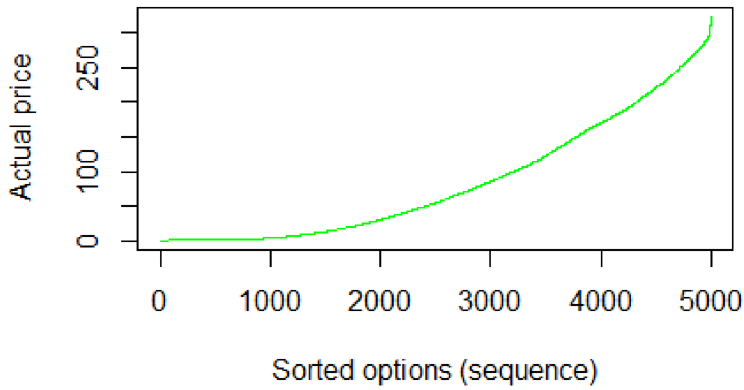

Figure 5. Sorted actual prices of options

5.1. Classical Black-Scholes model estimation results

Estimation results obtained from applying Black-Scholes model to the S&P500 index options data can be observed on the next Figure 6.

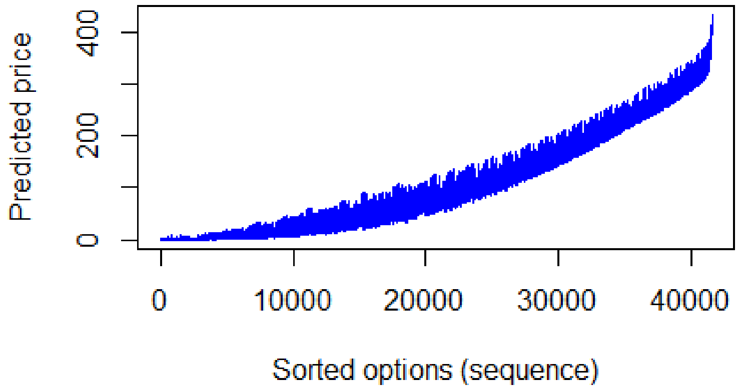

Figure 6. Classical Black-Scholes pricing errors

From the graph we can see that the classical BS model poorly (in terms of error metrics described above) estimates options with low prices. The main drawbacks of the model result into significant mispricing of the options, because of normality and stationary data (constant volatility of underlying asset prices is possible only under the assumption of stationary price series) assumed by the model.

From the Tables 4 and 5 we can observe errors of the model in more details. We can see that the worst results in terms of MAPE appear for the options out-of-the money. However, as we saw from the Table 3, prices for those options are extremely low, so the mispricing is actually possible.

Table 4. Mean absolute percentage errors of Black-Scholes model:

money-ness / matu-rity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9 - 1	1-1.05	1.05-1.1	1.1-1.5	Totals
1-40	0	0.01	0.01	0.01	0.02	0.1	2.02	2.92	1.19	**0.81**
41-90	0.01	0.01	0.02	0.03	0.03	0.1	0.56	2.63	1.84	**0.52**
101-250	0.02	0.04	0.05	0.06	0.06	0.11	0.28	1	4.04	**0.74**
251-500	0.05	0.07	0.08	0.09	0.1	0.12	0.19	0.35	1.28	**0.29**
>500	0.1	0.11	0.13	0.13	0.14	0.16	0.19	0.27	0.53	**0.22**
Totals	**0.04**	**0.05**	**0.05**	**0.06**	**0.05**	**0.11**	**0.93**	**1.78**	**1.98**	**0.56**

Table 5. Root mean squared errors of Black-Scholes model

moneyness / maturity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9 -1	1-1.05	1.05-1.1	1.1-1.5	To-tals
1-40	2.2	5.7	15.1	27.0	46.4	21.8	2.2	5.7	15.1	27.0
41-90	2.2	5.3	13.1	21.5	35.0	15.8	2.2	5.3	13.1	21.5
101-250	1.9	4.0	9.7	15.1	24.8	10.5	1.9	4.0	9.7	15.1
251-500	2.5	5.3	8.6	11.2	15.8	7.2	2.5	5.3	8.6	11.2
>500	4.2	9.8	16.2	18.7	18.0	11.2	4.2	9.8	16.2	18.7
Totals	4.3	10.4	18.9	25.5	24.1	13.3	4.3	10.4	18.9	25.5

5.2 SABR model estimation results

To estimate SABR model, we firstly need to calculate the implied volatility. This procedure can easily be done by using inverse Black-Scholes formula and find such a volatility, with which the model will fit to the actual price most accurately. This procedure is done using R package RquantLib. Distribution of implied volatilities in comparison with volatilities calculated with the sliding window approach can be seen in the next figures:

Table 6. Implied volatility

moneyness / maturity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9-1	1-1.05	1.05-1.1	1.1-1.5	Totals
1-40	0.22	0.2	0.21	0.22	0.22	0.2	0.2	0.18	0.2	**0.2**
41-90	0.1	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.1	**0.09**
101-250	0.1	0.09	0.09	0.1	0.1	0.1	0.1	0.09	0.1	**0.1**
251-500	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	**0.08**
>500	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	**0.07**
Totals	**0.11**	**0.11**	**0.12**	**0.12**	**0.14**	**0.14**	**0.13**	**0.12**	**0.1**	**0.12**

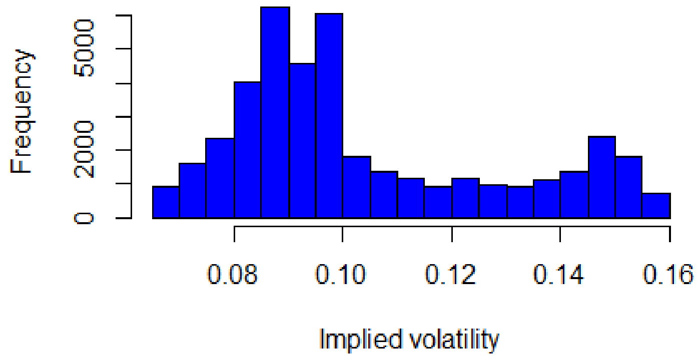

Figure 7. Implied volatility calculated using inverse BS formula

Figure 9 provide an insight about accuracy of the SABR model built.

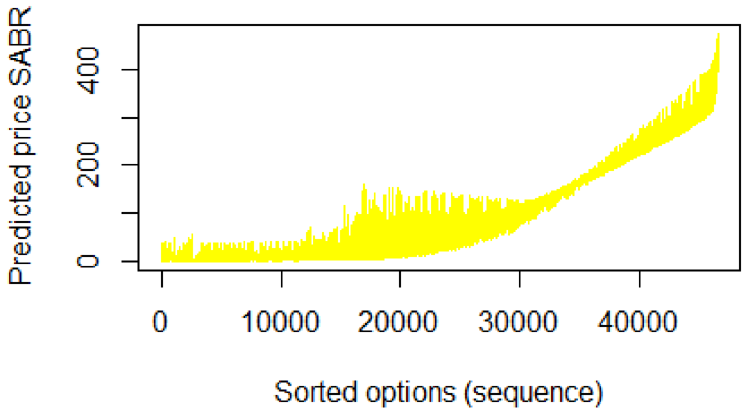

Figure 8. SABR model pricing errors

From the Figure 8 we can see, that SABR misprices option with low price. In the Tables 7 and 8 we can observe errors of the model in more details. Errors for out-of-the-money options are smaller than those for Black-Sholes model. Looking into rot mean squared errors, we can see that options which moneyness more then one, are mispriced in both models, hoverer SABR model gives more precise estimates (in terms of both mean absolute percentage errors and root mean squared errors).

Table 7. Mean absolute percentage errors of SABR model

moneyness / maturity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9 -1	1-1.05	1.05-1.1	1.1-1.5	totals
1-40	0	0	0.01	0.01	0.01	0.14	0.99	1	1	**0.4**
41-90	0.01	0.01	0.02	0.02	0.02	0.22	0.96	1	1	**0.44**
101-250	0.02	0.04	0.05	0.05	0.03	0.18	0.83	1	1	**0.37**
251-500	0.05	0.06	0.08	0.07	0.04	0.12	0.55	0.98	1	**0.36**
>500	0.1	0.11	0.12	0.11	0.08	0.07	0.22	0.67	0.99	**0.39**
Totals	**0.03**	**0.04**	**0.04**	**0.04**	**0.02**	**0.16**	**0.88**	**0.97**	**1**	**0.39**

Table 8. Root mean squared errors of SABR model

moneyness / maturity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9 -1	1-1.05	1.05-1.1	1.1-1.5	Totals
1-40	1.9	2.2	2.2	2.0	1.5	4.4	5.3	0.3	0.2	3.3
41-90	5.8	5.7	5.6	4.7	2.2	9.1	13.7	2.4	0.3	7.9
101-250	15.1	15.5	14.8	10.6	4.1	12.6	26.3	14.5	2.7	13.4
251-500	28.0	28.2	25.0	17.5	6.8	12.9	32.3	34.9	11.7	22.3
>500	53.3	48.1	41.9	28.2	16.9	11.5	23.8	46.0	29.7	34.1
Totals	21.6	23.3	18.3	12.5	5.2	8.8	16.9	19.2	13.3	18.4

5.3 Fractional Black-Scholes model estimation results

Fractional Black-Scholes model is assumed to better fit non-stationary returns data, when returns are not independent. The results of the estimation of fractional Black-Scholes model show good accuracy and can be seen on Figure 9.

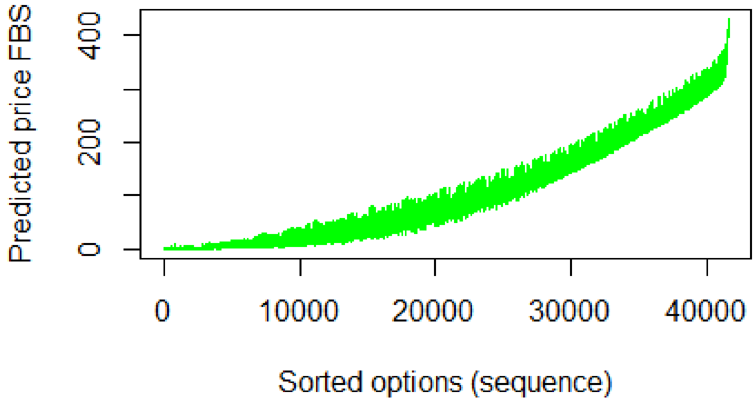

Figure 9. Fractional Black-Scholes model pricing errors

We can see that additional fractional parameters substantially reduce the error of the fit to the data.

Tables 9 and 10 contain more detailed information on pricing errors of FBS model. In particular, the model provides a very good fit to the prices of short- and medium-term and in-the-money options, but is less successful with longer-term and out-of-the-money contracts. If we compare these results with those of the classical BS model (Tables 4 and 5), we can notice that FBS is actually doing a worse job with longer-term and out-of-the-money options. We observe slightly better results in terms of mean absolute percentage error for all groups of options.

Table 9. Mean absolute percentage errors of Fractional Black-Scholes model

moneyness / maturity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9 - 1	1-1.05	1.05-1.1	1.1-1.5	To-tals
1-40	0.00	0.00	0.01	0.01	0.01	0.13	0.93	1.00	1.00	**0.38**
41-90	0.01	0.01	0.02	0.02	0.02	0.20	0.84	0.98	1.00	**0.41**
101-250	0.02	0.04	0.05	0.05	0.03	0.17	0.70	0.93	0.99	**0.35**
251-500	0.05	0.06	0.08	0.07	0.04	0.11	0.44	0.82	0.96	**0.33**
>500	0.10	0.11	0.12	0.11	0.08	0.07	0.19	0.50	0.88	**0.35**
Totals	**0.03**	**0.04**	**0.04**	**0.04**	**0.02**	**0.15**	**0.79**	**0.92**	**0.97**	**0.37**

Table 10. Root mean squared errors of Fractional Black-Scholes model

money-ness / maturity	0.3-0.6	0.6-0.7	0.7-0.8	0.8-0.9	0.9-0.95	0.9 - 1	1-1.05	1.05-1.1	1.1-1.5	Totals
1-40	2.24	2.49	16.25	2.35	1.98	27.22	20.27	1.42	0.22	15.86
41-90	5.69	5.87	5.85	5.03	3.59	5.23	6.07	3.99	0.99	5.10
101-250	15.02	15.57	14.66	11.73	8.38	9.27	10.53	10.36	6.19	11.61
251-500	29.37	29.28	26.45	20.73	15.06	13.57	14.51	14.91	12.41	20.73
>500	55.06	49.73	44.13	32.86	26.03	23.02	21.22	21.76	18.92	33.42
Totals	25.99	26.34	22.42	16.00	10.60	18.56	14.93	10.65	10.90	17.53

Estimated H parameter for the FBS is 0.69, which proves a long-time dependence of underlying S&P 500 index and coincides with results in Bayraktar et al (2003).

Q statistics is 0,997 and p-value of test is 0.043, so we failed to reject the hypothesis about the fractional nature of the underlying price of index.

SABR provides the worse in-sample fit than FBS model, and is comparable to the BS model in terms of root mean square errors and mean absolute percentage error

In sample errors can be summarized in the following table:

Table 11. Summary of accuracy of fit of the models

Model	RMSE	MAPE
Black-Scholes	25.5	0.56
Fractional Black Scholes	17.53	0.37
SABR	18.4	0.39

As we can see from the graphs, Fractional Black-Scholes provides more accurate results, especially in the middle range of option prices

Chapter 6

CONCLUSIONS

The purpose of this thesis is to estimate and compare empirical performance of three alternative option pricing models: classical Black-Scholes model, fractional Black Scholes model and Stochastic alpha, betha, rho model.

S&P 500 index options data is used to estimate option pricing models, predict a price according to each model and compare it to the actual. In order to put three models into the common estimation framework, volatility parameters are calculated in order to minimize the squared error between implied volatility parameters and computed volatilities. Black-Scholes model is then estimated as a baseline for the current study.

Next step is calculation of Hurst parameter for the Fractal Black-Scholes model and verification of the significance of the parameter obtained. Based on volatility parameter and Hurst parameter, FBS model is estimated.

Stochastic Alpha, Betha, Rho model is estimated using numerical Newton-Rhapson algorithm, modeled option volatility is then plugged into the original Black-Scholes model.

Accuracy metrics of models are obtained in terms of standard error metrics – root mean squared errors and mean average percentage errors of models, which are widely used for model comparison.

According to the results, S&P 500 option prices are most accurately predicted by the Fractional Black Scholes model. Performed statistical test suggest that we can use fractional model for these options. Fractional Hurst parameter obtained is consistent with the literature and reflects the behavior of the returns.

Fractional Black-Scholes model gives the best accuracy in the range of in-the-money and at-the-money option prices, and average accuracy for out-of-the-money options, however, the results are better than Stochastic Alpha Betha Rho and Black-Scholes models.

Conclusions are, that emerging model of Fractional Black-Scholes model is a next step in the evolution of the classical approach to the modeling, with better heuristics and more intuitively explained parameters, by the same time with competitive accuracy of the model.

WORKS CITED

Alexander, Carol, and Leonardo M. Nogueira. 2007. "Model-free price hedge ratios for homogeneous claims on tradable assets." *Quantitative Finance* 7, no. 5: 473-479.

Bartlett, Bruce. 2006. "Hedging under SABR model." *Wilmott magazine* 4: 2-4.

Bayraktar, Erhan, Hudson V. Poor and Kerrol R. Sircar. 2004. "Estimating the fractal dimension of the S&P500 index using wavelet analysis." *International Journal of Theoretical and Applied Finance* 07, no. 05: 615-643.

Cox, John C., and Stephen A. Ross. 1976. "The valuation of options for alternative stochastic processes." *Journal of Financial Economics* 3, no. 1-2: 145-166.

Flint, Emlyn, and Eben Maré. 2017. "Fractional Black–Scholes option pricing, volatility calibration and implied Hurst exponents in South African context." *South African Journal of Economic and Management Sciences* 20, no. 1.

Hagan, Patrick S., Deep Kumar, Andrew S. Lesniewski, and Diana E. Woodward. 2002. "Managing smile risk." *The Best of Wilmott* 1: 249-296.

Heston, Steven L. 1993. "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options." *Review of Financial Studies* 6, no. 2: 327-343.

Higuchi, T. 1988. "Approach to an irregular time series on the basis of the fractal theory." *Physica D: Nonlinear Phenomena* 31, no. 2: 277-283.

Hu, Yaozhong, and Bernt Oksendal. 2003. "Fractional white noise calculus and application to finance." *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 06, no. 01: 1-32.

Jarrow, Robert A., Philip Protter, and Kazuhiro Shimbo. 2010. "Asset price bubbles in incomplete markets." *Mathematical Finance* 20, no. 2: 145-185.

Le, Truc. 2015. "Efficient Markets Hypothesis: A False Prophesy. Black-Scholes-Merton Formula: A Parlor Trick. Risk-Neutral Pricing Models: Severe Malpractices." *SSRN Electronic Journal*.

Li, Kinry Q., and Rong Chen. 2014. "Implied Hurst Exponent and Fractional Implied Volatility: A Variance Term Structure Model." *SSRN Electronic Journal*.

Mandelbrot, Benoit, and Richard L. Hudson. 2007. *The Misbehavior of Markets: A fractal view of financial turbulence*. Basic books.

Mandelbrot, Benoit B., and James R. Wallis. 1969. "Computer Experiments with Fractional Gaussian Noises: Part 2, Rescaled Ranges and Spectra." *Water Resources Research* 5, no. 1: 242-259.

Mills, K., M. Vinson, and G. Cheng. 1992. "A large scale comparison of option pricing models with historical market data." *The Fourth Symposium on the Frontiers of Massively Parallel Computation*.

Mišhura, Julija S. 2008. "Stochastic Calculus for Fractional Brownian Motion and Related Processes." Berlin, Springer.

Necula, Ciprian. 2002. "Option Pricing in a Fractional Brownian Motion Environment." *SSRN Electronic Journal*.

Nourdin, Ivan, David Nualart, and Ciprian A. Tudor. 2010. "Central and non-central limit theorems for weighted power variations of fractional Brownian motion." *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* 46, no. 4: 1055-1079.

Peters, Edgar E. 1994. "Fractal Market Analysis: Applying Chaos Theory to Investment and Economics." New York: J. Wiley.

Ray, Sarbapriya. 2012. "A close look into Black-Scholes option pricing model." *Journal of Science* 2, no. 4: 172-178.

Rodriguez Longarela, I., and Silvia Mayoral. 2007. "A Theory of Inefficient Quotes: Empirical Evidence in Options Markets." *SSRN Electronic Journal.*

Taqqu, Murad S., Vadim Teverovsk, and Walter Willinger. 1995. "Estimators for long-range dependence: an empirical study." *Fractals* 03, no. 04: 785-798.

West, Graeme. 2005. "Calibration of the SABR Model in Illiquid Markets." *Applied Mathematical Finance* 12, no. 4: 371-385.

APPENDIX

RESCALED RANGE ANALYSIS

Algorithm of R/S analysis is following (Peters 1994):

1. Begin with a time series of length M. Convert it into a time series of length N = M - 1 from logarithmic equations.

$$N_i = \log\left(\frac{M_{i+1}}{M_i}\right), i = 1,2,3\ldots,(M-1). \tag{25}$$

or using the AR (1)-differences.

2. Divide this period of time by A adjacent subperiods of length n, so that A*n=N. Label each subperiod I_a, given that a = 1, 2, 3…, A. Each element in I_a is labeled N_k with k = 1, 2 , 3…, N. For each I_a of length n the mean value is defined as:

$$E_a = (I/n)\sum_{k=1}^{n} N_{k,a} \ , \tag{16}$$

where E_a is the mean value of N_k contained in subperiod I_a of length n.

3. Time series of accumulated deviations $(X_{k,a})$ on the mean value for every subperiod I_a is defined as:

$$X_{k,a} = \sum_{i=1}^{k}(N_{i,a} - e_a), k = 1,2\ldots,n. \tag{27}$$

4. The range is defined as the maximum value minus the minimum value $X_{k,a}$ within each subperiod I_a:

$$R_{k,s} = \max(X_{k,a}) - \min(X_{k,a}), k = 1,2,3,\ldots,n. \tag{28}$$

5. The sample standard deviation is calculated for each subperiod I_a:

$$S_{ia} = \left(\frac{l}{n} \cdot \sum_{k=1}^{n} \left(N_{k,a} - e_a^2 \right) \right)^{0.5}. \quad (29)$$

6. Each band R_{ka} is now normalized by dividing by the matching S_{ia}. Therefore, re-rescaled range during each subperiod I_a equals R_{ia}/S_{ia}. In step 2 above, we got adjacent subperiods of length n. Consequently, the mean value of R/S of length n is defined as:

$$(R/S)_n = \frac{1}{A} \cdot \sum_{a=i}^{A} \left(R_i / S_{i,a} \right) \quad (30)$$

7. The length n is increased to the next higher value, and $(M - I)/n$ is an integer value. We use the value n, including the start and end points of a time series, and steps 1-6 repeated until $n = (M - 1)/2$. We can perform a simple linear regression on log(n) as an independent variable and on $\log(r/s)_n$ as a dependent variable. The line segment cut off by a coordinate axis is an estimate of log(c), a constant. The slope of the equation is an estimate of the Hurst exponent.

Next step is further calibration of the parameter obtained. The process explained by Nourdin et al (2010), requires calculating of special statistics Q

$$Q = \frac{0,8}{R_1} \sqrt{\frac{(S_H^{-1}\mathbf{z}, \mathbf{z})}{n-1}} \quad (31)$$

where S_H - normalized correlation matrix of fractional Brownian motion increments,

$z = (z_1, \dots, z_{n-1})$. – vector of the realizations of fBm (log-differenced price of asset).

Elements of the correlation matrix S_H are calculated using the next formula

$$\mathrm{s_{ij}} = \frac{(j-i-1)^{2H} + (j-i+1)^{2H}}{2} + (j-i)^{2H} \quad (32)$$

Q statistics is evaluated for different values of H with some step, and the best value of the parameter H corresponds to the value of statistics closer to 1.

$$H = argmin\,|1 - Q\,(H)| \tag{33}$$

Using the simple numerical algorithm, which gives more accurate estimation, but demands a lot of computation facilities, we arrive at the final value of the computed Hurst parameter.

Final step is to verify statistical significance of the H parameter obtained.

Firstly, one needs to calculate

$$\upsilon_k = \sum_{j=1}^{k-1} z_j \tag{34}$$

$$\text{And } v_k^2 = \sum_{j=1}^{k-1} z_j^2 \tag{35}$$

and use statistics

$$B_n = \frac{1}{n^{1+H}} \sum \upsilon_k^2 z_k^3 \tag{36}$$

if $H \epsilon (0; \frac{1}{2})$ or

$$D_n = \frac{1}{n^{2H}} \sum \upsilon_k z_k^3 \tag{37}$$

If $H \epsilon (\frac{1}{2}; 1)$.

Where z_k are log- transformed prices of the asset.

If the hypothesis about the fractional Brownian motion with the obtained Hurst parameter is valid for the underlying process , and price differences belong to fBm according to its definition, then the next statement should be valid

$$A_n \to A = -\frac{3}{2}c^2 \qquad (38)$$

$$B_n \to 3c^{\frac{5}{2}} \qquad (39)$$

$$D_n \to \frac{3}{2}c^2 B^2(1) \qquad (40)$$

where $c = \frac{1}{n}\sum_{k=1}^{n} z_k^2$, and random variable $\eta \sim N(0; \frac{1}{2H+2})$.

To verify the hypothesis one needs to verify the next condition. The hypothesis about the fractional process fails to be rejected if inequality is valid:

$$0 < D_n < \beta_2 \qquad (41)$$

if H> 0,5 or

$$|B_n| < \beta_1 \qquad (42)$$

if H <0,5,

where B_k- quantiles of limit distributions, corresponding to the significance level α.

If $\alpha = 0{,}1$ then

$$\beta_1 = \frac{4{,}95c^{2{,}5}}{\sqrt{2H+2}} \qquad (43)$$

$$\beta_2 = 4{,}08c^2 \qquad (44)$$