On the Role of Incomplete Information in Investment Projects
MA Thesis

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Abstract

The sensitivity of the equilibrium set to the information structure of the game is widely studied. Nevertheless there is a broad range of the economic models where the role of the complete information assumption was not completely understood. In this paper I investigate the role that incomplete information plays in the modeling of the particular strategic investment decisions.

I consider a model of a private provision of the public goods with nonconvex technology under the assumptions of common and almost common knowledge. Applying a global game approach I demonstrate the existence of the unique Nash equilibrium, what is in contrast to the usual multiplicity of equilibria in the similar models.

In the model of a sequential investments project it is demonstrated how severe agency problems may arise from the uncertainty about the agents' types. The optimal wage schedule used to mitigate such problems is also developed.

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Acknowledgments

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Moreover Author wisely thanks all those who think that Author should think that he should be grateful to for some more or less evident reasons. Also, recognizing the complete incompleteness of the previous statement, Author thanks all those who think that Author should think that they should think that Author should be grateful to them for some even more mysterious but still unquestionable reasons. Anticipating the need for nontrivial mental efforts of going even further in his appreciation, Author generously grants the right to anybody who for whatever reasons wishes to be appreciated by Author to do it by themselves in the name Author.

1 Almost Common Knowledge in the Game of Private Provision of Public Good with Nonconvex Technology

1.1 Introduction

The standard models that consider the private provision of public goods often incorporate a property of the so-called *self-fulfilling beliefs*. Roughly speaking it means that beliefs of economic agents about the behavior of other agents are considered exogenous and so are left outside the scope of economic analysis. Taking into account the significance of a coordination in voluntary provision of public goods by separate private agents, given feature substantially weakens the explanatory and predictive power of these models through generating the multiplicity of equilibria for certain values of parameters. In this paper I develop incomplete information model of private provision of public goods with *almost common knowledge* of production technology. I manage to demonstrate how naturally relaxing the assumption of common knowledge lead to endogenous belief formation and to a simple unique equilibrium.

Consider the technology of public good production that is characterized by nonzero fixed cost. Consumers simultaneously decide whether to invest or not to invest in the production of public good. If the amount of contributions is less than fixed costs public good is not produced and money are wasted. Andreoni shows [6] that under quite general conditions this model will have multiple equilibria – one when enough consumers invest and public good is produced and another when no one invests and public good is not produced. I develop the similar theoretical framework and consider both – case with complete and incomplete information about production technology. While the multiplicity of Nash equilibria still arises in the model with complete information, I, on the contrary, prove that if agents are not perfectly informed about the productivity of a particular technology and rather get imperfect correlated private signals, there will exist the unique threshold Nash equilibrium under which consumer contributes only if his signal is greater than some cutoff value. Thus my result points out the problem of oversimplification in previous models of private provision of public goods with non-convex technologies. The multiplicity of equilibria in that models stemmed from the unrealistic assumptions of complete information and thus exogenous beliefs.

The intuition behind my result lies in the fact that when the private signals on technology incorporate some arbitrary small amount of noise, then the player's beliefs about other's players beliefs are no longer a common knowledge. Even if all the players know the value of a particular parameter and moreover all of them also know that all know this, it may not be true that the player knows that the other player knows that he knows this. So the *higher order beliefs*, that is beliefs about beliefs about payoffs substantially modify the analysis and requires that the equilibrium strategies are globally consistent – that is accord with all the possible beliefs that the players may hold. Such way of reasoning was first developed by Carlsson and van Damme [9] and is called in the literature the *global game* approach.

1.2 Literature Review

The seminal paper in the modern theory of private provision of public goods is that of Bergstrom, Blume and Varian (BBV) [8]. They demonstrate, departing from the earlier paper of Warr [31], the existence of Nash equilibrium with positive provision in a basic model and generalize the neutrality theorem[1] under quite weak assumptions. The model of this type normally assumes the so-called purely altruistic preferences of consumers. That is consumers care only about their consumption of private good (x_i) and the *total* production of public good (G): $u = U(x_i, G)$. With standard assumptions on preferences they demonstrate that in Nash equilibrium in general there will be nonempty set of contributors to public good. Note that technology is assumed to be linear in total sum of contributions. Paper by BBV laid a clear and precise theory behind public goods provision and charity giving and so it was widely extended in the subsequent literature[2]

Nevertheless as it was demonstrated by Andreoni [4] pure altruism model appears to be not sufficient in explaining certain empirical regularities. For example theory predicts that with the economic growth the fraction of contributors must go to zero what is in sharp contrast to stylized facts about charity. Thus non-altruistic motives should be considered in order to create realistic theory of giving. One extension is to change preferences in a way

[1] The theorem basically states that after the redistribution of wealth among consumers they consume the same amount of public and private goods that they did before the redistribution

[2] See e.g. special issue of Journal of Public Economics for review and discussion of current topics in the area – Journal of Public Economics 91 (2007) 1643 - 1644.

that utility directly incorporates agent's own contribution to public good: $u = U(x_i, G, g_i)$. Intuitively this is done to capture the "warm-glow effect" from giving – when consumer experience additional satisfaction from having contributed to the public good. Theory of "warm-glow" giving is in detail discussed in Andreoni [5]. In particular he concludes that impurely altruistic preferences may help to explain incomplete crowding out of private donations by governmental financing.

While much of the previous theory does not elaborate on the technology of producing a public good, Andreoni [6] considers an important case of non-convex technologies (as e.g. in the case of positive fixed costs of production). He incorporates technology with fixed costs into the standard model and shows that this results in existence of zero contributions equilibrium along with a positive contributions equilibrium (with contributions equal to equilibrium ones in a model without fixed costs). Roughly speaking, if there is no such consumer who can solely contribute the whole amount of fixed costs needed to produce a public good then there will exist an equilibrium in zero. It is worth mentioning that if there is a consumer who can contribute fixed costs alone (and so provide public good at at least minimum possible level while still being better off) then it is no need for him to do so. Other agents also will contribute and so will reduce the optimal amount of contribution by that consumer. Andreoni uses the fact of the existence of zero equilibrium to explain the wide spread practice among professional fund-risers not to start public fund-rising campaign until substantial amount of initially needed sum is not collected as the so-called "seed money" from the leading donors. He shows that the existence of such "seed money" may eliminate an undesirable zero equilibrium (when no one actually contributes money) and so rise the expectation of collected funds.

The arguments concerning the role of a quality of a charity or a public campaign are also developed in the literature (notice that this is basically equivalent to consideration of the productivity - a quality of technology of production of public good). Here the signalling role of leadership giving is emphasized. Leader can bear costs of investigating the true quality of campaign and thus his participation is a signal to others to invest. Conclusion of such model is that the exceptionally high gifts can be demanded from the leader in order for his signals to be credible [3].

As I have mentioned earlier, I modify the information structure of the public good provision model relaxing the assumption of common knowledge. So in general I apply the global game analysis to the coordination problem game between consumers who decide on there contributions to public good. Global games provide a natural and thus attractive equilibrium selection basis. It is especially evident when one considers the games with the strategic complementarities such as the games of private provision of public good or the models of a charity donations. The issue is that in such framework decisions of consumers to contribute to charity or to a construction of public goods are natural complements what is even more important when production technology is non-convex as in the case with nonzero fixed costs.

Carlsson and van Damme [9] are the first who systematically treat global games as the incomplete information games where the actual game played is randomly drawn from some class of games and players get noisy signals on the game chosen. They consider simple 2x2 coordination game and show how given information structure forces equilibrium selection consistent with a risk dominance criteria. They use iterated dominance arguments to establish a unique risk dominant equilibrium in a general class of 2x2 games.

Essentially, the notion of a global game is closely related to that of a common knowledge (first formalized by Aumann [2]). In a global game beside the fact that there is a common knowledge of the prior distributions and also of signal-generating techniques there is no common knowledge of beliefs about the beliefs about the beliefs etc. The illustrative example with imperfect private signals called "The Electronic Mail Game" was presented by Rubinstein [27]. There are two players, one of them observes the state of the world. Then he automatically sends a message about the state to other player, who also automatically responds if he gets the message and so on. Rubinstein points out on counterintuitive conclusion that in Nash equilibrium no matter how much messages were sent no one will play risky action that demands coordination from both players.

The idea of global games was fruitfully applied in many economic models in the presence of strategic complementarities. Morris and Shin [25] demonstrate how global games approach guarantees the unique equilibrium in a model of currency attacks with a continuum of traders deciding to attack the fixed exchange rate or not. Attack is successful if large enough proportion of traders decide to attack. Traders get private signals on the value of economic fundamentals and in the equilibrium there is one such value above which currency attack always happens.

A broad summary of applications of global games framework to macroeconomic models is given by Morris and Shin [24]. In particular they also consider models of bank runs and pricing of debt. Generally, the authors claim that shift of beliefs that result in selection between equilibrium outcomes is left unexplained in the standard economic models. They argue that "sunspot" explanations based on the logic of self-fulfilling beliefs do not provide a link between economic fundamentals and economic outcomes and thus there is a need for an reexamination of theoretical basis for multiple equilibria.

1.3 Model with a Common Knowledge of a Quality of Investment Project

Consider two players (consumers) that simultaneously decide whether to invest in the construction of public good[3]. Such good may be for example a new bridge, kindergarten building etc. The first stage of the game can be thought as the stage when the public donations campaign is launched and voluntary contributions are collected. At this stage the players take a decision. At the second stage, when the campaign is over and public good is either produced (if sufficient amount of funds was collected) or not, players consume the good (if produced) and payoffs are realized. Production technology is characterized by *non-zero fixed costs*. Obviously the production of already mentioned and many other public goods requires some minimum threshold amount of initial investments, that is there exist positive fixed costs. If amount of public contributions does not meet the threshold implied by fixed costs then public good is not build at all and collected money are wasted. Moreover the production technology itself matters a lot. This technology may be more or less effective, that is it may require less or more money to produce the same amount of a public good (of a constant quality). A productivity of the technology that is used is a quite general property. It can be also thought as the one, representing the quality of charity campaign – resources can be managed poorly or, on the contrary, with a great effectiveness. Better the campaign is run, the more effective is the use of collected funds – less money are needed to bring the same benefits to consumers.

Such technological process of producing a public good can be formally

[3]Note that the two-player game presented here can be easily generalized to a case with multiple players. The main idea and the analyzes remains essentially the same but the proofs become a bit more cumbersome.

summarized by the following production function:

$$G = \begin{cases} \alpha \sum_{i=1}^{2} g_i & \text{if } \alpha \sum_{i=1}^{2} g_i \geq \bar{G} \\ 0 & \text{if } \alpha \sum_{i=1}^{2} g_i < \bar{G} \end{cases}$$

where G is a total quantity of public good, g_i is the player's ith contribution to public good, $\alpha \in \mathbb{R}$ is a productivity parameter[4] and $\bar{G} > 0$ is a fixed cost of production.

Players can contribute fixed amount of money $c > 0$ or not contribute at all, that is action of player is $g_i \in \{c, 0\}$.

Payoff of a consumer depends on the total amount of a good produced and his decision. Also note that consumer can consume a good when it is produced even if he did not contribute at the production stage (the good is, basically, public – e.g. everyone can use a bridge): $U_i = U(G) - g_i$. Players are homogeneous and for simplicity I assume that the function U takes the following form:

$$U = \begin{cases} \alpha\bar{U} & \text{if } G = \alpha c \\ 2\alpha\bar{U} & \text{if } G = 2\alpha c \\ 0 & \text{if } G = 0 \end{cases}$$

Where $\bar{U} > 0$. So the safe option not to contribute guarantees a consumer payoff equal to, at least, zero.

Consider the lower bound on the productivity of the technology α that still permits the construction of public good with the contribution from the only one consumer. That is the lowest α for which c is enough to build a public good. It is given by $\alpha c = \bar{G}$, so let $\hat{\alpha} := \frac{\bar{G}}{c}$ be such lower bound. If $\alpha \geq \hat{\alpha}$ then public good can be build solely by one contributing consumer. What means that the technology is extremely productive or a public donations campaign is of superior organization.

Similarly define lower bound on α under which public good can not be build even if both consumers contribute amount c. It is given by $2\alpha c = \bar{G}$ and so let $\breve{\alpha} := \frac{\bar{G}}{2c}$ be such bound. Values of $\alpha < \breve{\alpha}$ correspond to extremely bad technology of producing a public good. Obviously $\hat{\alpha} > \breve{\alpha}$.

I will consider the values of $\bar{U}$ high enough to ensure that they imply positive payoff from consuming public good after paying the contribution $c > 0$ even if the other player did not invested and α is sufficiently low. That

[4] I allow for negative values of α in order to simplify subsequent analysis. This assumption is not crucial to our main result.

is consumer is willing to invest if other one invests for all $\alpha \geq \check{\alpha}$ and even if the other one does not for all $\alpha \geq \hat{\alpha}$. This imply the following inequalities:

$$\hat{\alpha} \geq c$$
$$2\check{\alpha} \geq c$$

From what, given values of $\hat{\alpha}$ and $\check{\alpha}$, immediately follows $\bar{U} \geq \frac{c^2}{G}$. Such assumption is purely technical and is needed to ensure that individual rationality constraint does not bind in equilibrium, so we can concentrate solely on the primary issues.

For now we are assuming that players are *perfectly informed* about the value of α and it it is a common knowledge. So they now exactly of what quality is a particular public donations campaign or put in other words they perfectly observe the technology of producing the public good. And every player also knows that the other one knows this and so on.

We look for a Nash Equilibrium of this complete information game. The following theorem is immediately evident.

Theorem 1 *Depending on the value of α there are following Nash equilibrium strategy profiles in a given game:*

1. $\langle c, c \rangle$ if $\alpha > \hat{\alpha}$
2. $\langle 0, 0 \rangle$ if $\alpha < \check{\alpha}$
3. $\langle 0, 0 \rangle$, $\langle c, c \rangle$ if $\alpha \in [\check{\alpha}, \hat{\alpha}]$

Note the multiplicity of equilibria when $\alpha \in [\check{\alpha}, \hat{\alpha}]$. It is a common feature of a model of private provision of public good in the presence of fixed costs of production (as in the model of Andreoni [6]). As far as the most interesting case is when the technology is not on the extremes but somewhere in between the bounds for productivity parameter such multiplicity of equilibria can not be satisfactory. Roughly speaking it is hardly possible to infer particular behavior prediction from such a model of private provision of public goods. It is not clear what outcome to expect and there is no any other naturally acceptable and intuitive way to discriminate between two equilibria. But it appears that such problem is merely a problem of oversimplification in a baseline model. In the next section I show how naturally and realistically relaxing the assumption of complete information leads to simple unique equilibrium in game of a private provision of public goods with nonconvex technology.

1.4 Model without Common Knowledge of a Quality of Investment Project

Now I drop the assumption of complete information and in particular of common knowledge in a given game. Suppose that consumers can not perfectly observe the technology that is used to produced a public good. It may be too costly for a single consumer to investigate the business plan of a public donations campaign, examine the specific features of the the project etc. All information on technology they get is the private signal on productivity α. This information may come from different sources for different consumers such as rumors, their previous experience and competence. I assume that initially all the values of α are equally likely[5]. Once a particular value is selected by nature in the beginning of the game each player obtains an imperfect private signal. This signal is given by:

$$x = \alpha + \epsilon, \quad \epsilon \sim N(0, \sigma^2), \text{ i.i.d. across players}$$

From what it is evident that conditional distribution of α will be normal with mean x and variance σ^2. After observing the private signal a player is interested in knowing the distribution of the signal of his opponent. He basically asks the question: what can I say about the signal of other player given that my signal is x? So, given his own signal every player infers conditional distribution of the signal of his opponent. Let x' be a signal of the opponent and ϵ' be it's error term, then from $Ex' = E\{x - \epsilon + \epsilon'\} = Ex$ and $Var\{x'\} = Var\{x - \epsilon + \epsilon'\} = 2\sigma^2$ we get that $x' \sim N(x, 2\sigma^2)$.

Notice that such information structure implies that player's beliefs about other's beliefs are no longer a common knowledge as it was discussed earlier.

In a search for the equilibrium the natural candidate will be a following switching strategy:

$$\pi(x) = \begin{cases} c & \text{if } x > \tau \\ 0 & \text{if } x \leq \tau \end{cases}$$

So the player invests if only if he gets a signal greater than a certain threshold value τ. The main result of this paper, presented below, states that such

[5]Such assumption means actually *improper (generalized)* prior distribution of the parameter – that is it has infinite mass. Nevertheless as far as only the posterior distribution is of primary interest it is common in the literature to make such assumptions (see e.g. [7]).

strategy is the only Nash equilibrium strategy in a given game of incomplete information.

Theorem 2 *There exists a unique τ such that neither player has an incentive to deviate from $\pi(x)$ so that it is a unique Nash Equilibrium of the given game.*

Proof I will prove the theorem in the four following steps:

1. Determine expected payoffs of the player from contributing and not contributing in the construction of public good.

2. Establish important properties of these expected payoff functions.

3. Define best response function using obtained expected payoffs and establish it's properties.

4. Demonstrate the existence and uniqueness of equilibrium in the model.

I begin from the construction of expected payoff function from investing in public good. Suppose other player plays strategy $\pi(x)$. Then the expected payoff of the first player *from investing* is given by:

$$\Phi(\frac{\tau - x}{\sqrt{2}\sigma})E_0\{U|x\} + \left(1 - \Phi(\frac{\tau - x}{\sqrt{2}\sigma})\right) E_c\{U|x\} - c$$

where $\Phi(\cdot)$ is a cumulative standard normal and thus $\Phi(\frac{\tau-x}{\sqrt{2}\sigma})$ is a probability that the second player got a signal less than τ and so does not invest. $1 - \Phi(\frac{\tau-x}{\sqrt{2}\sigma})$ is a complementary probability – a probability that the second player invests. Remember that conditional distribution of the other player's signal is normal with mean x and standard deviation $\sqrt{2}\sigma$. c is a cost of investing – it is paid for certain.

$E_0\{U|x\}$ is an expectation of U given signal x and that the other player does not invest (chooses $g = 0$):

$$E_0\{U|x\} = 0 \ Prob\{\alpha < \hat{\alpha}|x\} + E_0\{U|x, \alpha > \hat{\alpha}\} \ Prob\{\alpha > \hat{\alpha}|x\}$$

Taking expectation we get:

$$E_0\{U|x, \alpha > \hat{\alpha}\} = \frac{\bar{U}}{1 - \Phi(\frac{\hat{\alpha}-x}{\sigma})} \int_{\hat{\alpha}}^{\infty} \alpha f(\alpha) d\alpha$$

where $f(\alpha) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\alpha - x)^2}{2\sigma^2}}$ is a normal density function with mean x and standard deviation σ. So we obtain:

$$E_0\{U|x\} = \bar{U} \int_{\hat{\alpha}}^{\infty} \alpha f(\alpha) d\alpha$$

Let $\int_{\hat{\alpha}}^{\infty} \alpha f(\alpha) d\alpha := \Upsilon(x, \bar{\alpha})$.

Then differentiating $\Upsilon(x, \bar{\alpha})$ with respect to the mean x we get:

$$\frac{\partial \Upsilon(x, \hat{\alpha})}{\partial x} = \frac{1}{\sigma\sqrt{2\pi}} \int_{\hat{\alpha}}^{\infty} \frac{\alpha(\alpha - x)}{\sigma^2} \exp\left\{-\frac{(\alpha - x)^2}{2\sigma^2}\right\} d\alpha$$

Now I show that this derivative is always positive. The only unclear case is when $x > \hat{\alpha}$. Rewrite expression as:

$$\begin{aligned}
\frac{\partial \Upsilon(x, \hat{\alpha})}{\partial x} =& \frac{1}{\sigma\sqrt{2\pi}} \int_{\hat{\alpha}}^{x} \frac{\alpha(\alpha - x)}{\sigma^2} \exp\left\{-\frac{(\alpha - x)^2}{2\sigma^2}\right\} d\alpha + \\
&+ \frac{1}{\sigma\sqrt{2\pi}} \int_{x}^{x+\hat{\alpha}} \frac{\alpha(\alpha - x)}{\sigma^2} \exp\left\{-\frac{(\alpha - x)^2}{2\sigma^2}\right\} d\alpha + \\
&+ \frac{1}{\sigma\sqrt{2\pi}} \int_{x+\hat{\alpha}}^{\infty} \frac{\alpha(\alpha - x)}{\sigma^2} \exp\left\{-\frac{(\alpha - x)^2}{2\sigma^2}\right\} d\alpha
\end{aligned}$$

Note that the second term will be equal to the first term with the negative sign. Thus:

$$\frac{\partial \Upsilon(x, \hat{\alpha})}{\partial x} = \frac{1}{\sigma\sqrt{2\pi}} \int_{x+\hat{\alpha}}^{\infty} \frac{\alpha(\alpha - x)}{\sigma^2} \exp\left\{-\frac{(\alpha - x)^2}{2\sigma^2}\right\} d\alpha > 0$$

Let $\frac{\partial \Upsilon(x, \hat{\alpha})}{\partial x} := \gamma(x, \hat{\alpha})$.

Now consider $E_c\{U|x\}$. It is an expectation of U given x and that the other player invests. Again taking conditional expectation we get:

$$E_c\{U|x\} = 0\ \Phi(\frac{\check{\alpha} - x}{\sigma}) + 2\bar{U} \int_{\check{\alpha}}^{\infty} \alpha f(\alpha) d\alpha$$

Defining $\int_{\check{\alpha}}^{\infty} \alpha f(\alpha) d\alpha := \Upsilon(x, \check{\alpha})$ and $\frac{\partial \Upsilon(x, \check{\alpha})}{\partial x} := \gamma(x, \check{\alpha}) > 0$ we get the expected payoff from investing when the other player uses strategy $\pi(x)$ equal:

$$\Phi\left(\frac{\tau - x}{\sqrt{2}\sigma}\right) \bar{U} \Upsilon(x, \hat{\alpha}) + \left[1 - \Phi(\frac{\tau - x}{\sqrt{2}\sigma})\right] 2\bar{U} \Upsilon(x, \check{\alpha}) - c \qquad (1)$$

Consider the derivative of cumulative standard normal function with respect to signal x:

$$\frac{d\Phi(\frac{\tau-x}{\sqrt{2}\sigma})}{dx} = -\frac{\phi(\frac{\tau-x}{\sqrt{2}\sigma})}{\sqrt{2}\sigma}$$

where $\phi(\cdot)$ is a standard normal density function.

Now differentiating the expected payoff from investing with respect to x I obtain:

$$\bar{U}\left[\frac{\phi(\frac{\tau-x}{\sqrt{2}\sigma})}{\sqrt{2}\sigma}\left[2\Upsilon(x,\breve{\alpha})-\Upsilon(x,\hat{\alpha})\right]+\Phi(\frac{\tau-x}{\sqrt{2}\sigma})\gamma(x,\hat{\alpha})+2(1-\Phi(\frac{\tau-x}{\sqrt{2}\sigma}))\gamma(x,\breve{\alpha})\right]$$

Since:

$$2\Upsilon(x,\breve{\alpha})-\Upsilon(x,\hat{\alpha}) = 2\int_{\breve{\alpha}}^{\infty}\alpha f(\alpha)d\alpha - \int_{\hat{\alpha}}^{\infty}\alpha f(\alpha)d\alpha$$

and $\breve{\alpha} < \hat{\alpha}$ we get:

$$2\Upsilon(x,\breve{\alpha})-\Upsilon(x,\hat{\alpha}) > 0$$

We have already shown that $\gamma(x,\cdot) > 0$ thus the derivative of expected payoff from investing is positive $\forall x \in \mathbb{R}$. So, we can conclude, that the expected payoff from investing beside the obvious continuity is also strictly increasing in the obtained signal x.

Now I consider expected payoff from not investing. It is:

$$\left[1-\Phi(\frac{\tau-x}{\sqrt{2}\sigma})\right]\bar{U}\Upsilon(x,\hat{\alpha}) \tag{2}$$

The first term is the probability that the second player invests, the second term is the expectation of payoff of the first player given that the public good was provided. Note that expected payoff is bounded by zero from below. It reflects the fact that not contributing is the safe option – you can not loose money refraining from investing in the public good. Moreover you can gain from free-riding in case if the public good was financed solely by the other player. The expected payoff is also continuous, strictly increasing in x with the derivative equal:

$$\frac{\phi(\frac{\tau-x}{\sqrt{2}\sigma})}{\sqrt{2}\sigma}\bar{U}\Upsilon(x,\hat{\alpha}) + (1-\Phi(\frac{\tau-x}{\sqrt{2}\sigma}))\bar{U}\gamma(x,\hat{\alpha}) > 0$$

A player invests if only if the expected payoff from investing exceeds the expected payoff from refraining, given that the second player plays $\pi(x)$. These payoffs are plotted on the figure 1 below.

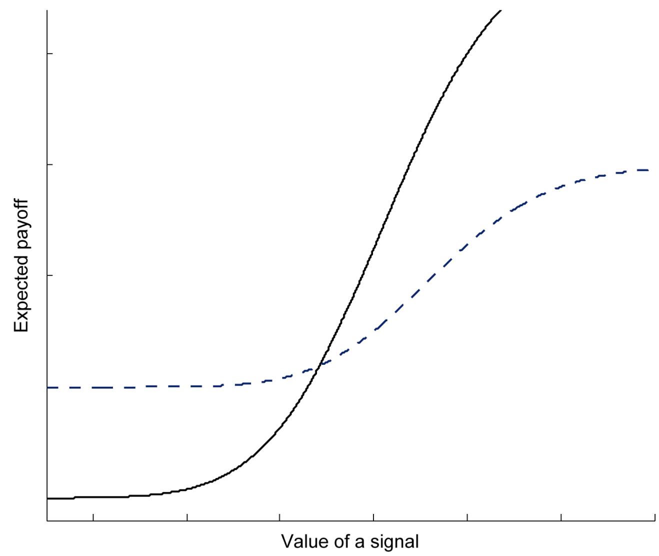

Figure 1: Determination of the best response to a particular threshold strategy

Consider the ratio of expected payoff from investing (1) and expected payoff from refraining (2):

$$\Delta(x) := \frac{\Phi\left(\frac{\tau-x}{\sqrt{2}\sigma}\right)\bar{U}\Upsilon(x,\hat{\alpha}) + \left[1-\Phi(\frac{\tau-x}{\sqrt{2}\sigma})\right]2\bar{U}\Upsilon(x,\check{\alpha}) - c}{\left[1-\Phi(\frac{\tau-x}{\sqrt{2}\sigma})\right]\bar{U}\Upsilon(x,\hat{\alpha})}$$

Let $x \to -\infty$. Then $\Phi(\frac{\tau-x}{\sqrt{2}\sigma}) \to 1$ (second player almost never invests) and $\Upsilon(x,\cdot) \to 0$ (since the probability that α is greater than some constant goes to zero). So, it follows that: $\lim_{x\to-\infty}\Delta(x) = -\infty$ (since denominator goes to zero and numerator goes to $-c$). That is expected payoff from refraining is greater than from investing.

Now let $x \to \infty$. Then $\Phi(\frac{\tau-x}{\sqrt{2}\sigma}) \to 0$, $\Upsilon(x,\cdot) \to \infty$. Obtain

$$\lim_{x\to\infty}\Delta(x) = \lim_{x\to\infty}\left(\frac{\Phi\frac{\tau-x}{\sqrt{2}\sigma}}{1-\Phi\frac{\tau-x}{\sqrt{2}\sigma}} + \frac{2\Upsilon(x,\check{\alpha})-c}{\Upsilon(x,\hat{\alpha})}\right) = 2$$

Since both payoffs are positive this means that the expected payoff from investing exceeds the expected payoff from refraining. Thus from continuity of both functions it follows that there always exists a point of intersection (that is when both payoffs are equal). Moreover one can notice that derivative with respect to x of the expected payoff from investing is always greater than the derivative of the expected payoff from refraining. That is the difference between them is always positive:

$$\bar{U}[2\frac{\phi(\frac{\tau-x}{\sqrt{2}\sigma})}{\sqrt{2}\sigma}[\Upsilon(x,\check{\alpha})-\Upsilon(x,\hat{\alpha})]+\Phi(\frac{\tau-x}{\sqrt{2}\sigma})\gamma(x,\hat{\alpha})+ \tag{3}$$

$$+\left(1-\Phi(\frac{\tau-x}{\sqrt{2}\sigma})\right)(2\gamma(x,\check{\alpha})-\gamma(x,\hat{\alpha}))]>0$$

This property insures the uniqueness of such point.

The point of intersection defines the unique threshold value of a signal for the first player that is the best response to the second player's switching strategy with threshold τ. If the signal is greater than this threshold value then expected payoff from investing is greater than expected payoff from refraining and the player should invest.

The best response function for the player is a function such that, given any threshold of an opponent, it returns the unique optimal threshold for the player. This function is implicitly defined by equalizing expressions for expected payoffs from investing (1) and not investing (2). Geometrically it means that given any value of τ the best response function maps it in the unique value of x such that this value corresponds to the intersection of the graphs of expected payoffs (as on the figure 1). Let $b:\mathbb{R}\to\mathbb{R}$ be this function implicitly given by equalizing difference between (1) and (2) to zero:

$$\Phi\left(\frac{\tau-b(\tau)}{\sqrt{2}\sigma}\right)\Upsilon(b(\tau),\hat{\alpha})+\left[1-\Phi(\frac{\tau-b(\tau)}{\sqrt{2}\sigma})\right][2\Upsilon(b(\tau),\check{\alpha})-\Upsilon(b(\tau),\hat{\alpha})]\equiv\frac{c}{\bar{U}}$$

The derivative of the left hand side of this expression is given by the difference in derivatives of expected payoffs (3) which was shown to exist and be positive for all $x\in\mathbb{R}$ and $\tau\in\mathbb{R}$. Thus by Implicit Function Theorem $b\in\mathbf{C}^1(\mathbb{R},\mathbb{R})$, that is map b belongs to a space of continuously differentiable maps from real line into real line (self-maps on $\mathbb{R}$).

By implicit differentiation it is clear that $\frac{db(\tau)}{d\tau}>0$ that is the function is monotone increasing. The intuition for this observation is that a greater τ of an opponent, that is the smaller probability that he invests, forces a player to be more pessimistic and to invest only for higher values of his own signal.

Now let $\tau \to \infty$ (that is the opponent almost never invests) then $b(\tau) \to \hat{x} \in \mathbb{R}$ and if $\tau \to -\infty$ (that is the opponent almost always invests) then $b(\tau) \to \check{x} \in \mathbb{R}$. It is easy to see that these bounds on values of x are implicitly given by respectively:

$$\Upsilon(\hat{x}, \hat{\alpha}) = \frac{c}{\bar{U}}$$
$$2\Upsilon(\check{x}, \check{\alpha}) - \Upsilon(\check{x}, \hat{\alpha}) = \frac{c}{\bar{U}}$$

What can be rewritten using the definition of $\Upsilon(\cdot)$ as:

$$\Upsilon(\hat{x}, \hat{\alpha}) = \frac{c}{\bar{U}}$$
$$2\int_{\check{\alpha}}^{\hat{\alpha}} \alpha f_{\check{x}}(\alpha) d\alpha + \Upsilon(\check{x}, \hat{\alpha}) = \frac{c}{\bar{U}}$$

Where $f_{\check{x}}(\alpha)$ is a normal density function with mean $\check{x}$ and variance σ^2. Since the left hand sides are strictly increasing in x and first term in the second equation is always positive, it follows that $\hat{x} > \check{x}$. Thus it is established that b ia also a bounded self-map on $\mathbb{R}$.

Now we are ready to come up with the primary result of existence of unique Nash Equilibrium of the game.

Consider the following self-map on $\mathbb{R}$: $(y_1, y_2) \to (b(y_2), b(y_1))$, $y \in \mathbb{R}$. This is the best response function of the game. But since the game is symmetric it is enough to demonstrate the existence of fixed point only for function b. Let $S := [\check{x}, \hat{x}]$ so it is nonempty, compact and convex subset of $\mathbb{R}$. Since b is a continuous self-map on S, then by Brouwer Fixed Point Theorem there exists an $y \in S$ such that $b(y) = y$. That is y is a unique fixed point of b. Thus the best response function has a fixed point which, actually, defines the unique Nash equilibrium in threshold strategy $\pi(x)$.

Q.E.D.

Notice that the result holds independently of the accuracy of private signal. As long as the noise component of a signal is nonzero – that is as long as $\sigma > 0$ – there will exist a unique threshold equilibrium. This means that even if players are *almost* perfectly informed about the actual technology used they still will play threshold strategies in equilibrium – invest if only if their private signal exceeds some cutoff value τ. This is the only strategy consistent with the full hierarchy of beliefs players may hold. Even if the noise is so small that both players almost for sure know the value of α, and, moreover, they almost for sure know that the other one knows this value too, they may not be sure that the fact that they know this is also known to the other player and so on. And thus the equilibrium strategies of complete information game with a common knowledge can not be the equilibrium strategies of the given game with a perturbed information structure.

Finally, presented here uniqueness result perfectly accords with the empirical evidence that public donations and charity campaigns can be successful even with no or with a little amount collected as "seed" money. The result in obvious way implies that for any given productivity parameter α we can calculate the expectation of total contributions. Moreover, since the change in α continuously change the means of respective distributions, this expectation appears to be continuous in α (besides that we do not present a formal proof of this argument it seems to be quite clear). So this explains the underlying mechanism under the conclusions of the natural experiment ran by List and Reiley [20]. Authors state that they found that seed money really rise the amount of collected contributions during the charity campaigns, but continuously and in a substantially lesser degree than that predicted by complete information models (basically these models predict the elimination of zero equilibrium and so the discontinuous jump in the contributions from zero to some positive amount [6]).

1.5 Conclusions

In this paper I develop a global game approach to a problem of private provision of public goods with nonconvex technology. Relaxing the assumption of complete information of the productivity of the technology used in production of a public good leads to a reduction of equilibrium set to the unique threshold Nash equilibrium. This is in the sharp contrast to a complete information version of the game with the multiplicity of equilibria. Thus it is shown that the models of public provision with nonconvex technologies based on complete information assumptions suffer from oversimplification and the multiplicity of equilibria found in this models is in of no way the relevant feature of the underlying real-world phenomena.

2 Costly Reputation: Application to Sequential Investment Project

2.1 Introduction

I develop a model of a sequential investment project to explain how reputational concerns of hired managers result in severe agency problem among them and investors, and what it actually costs investor to overcome such inefficiencies. An investor hires a manager and delegates him an authority to take certain decisions that are relevant to the success of a venture. If the investment project, as it is common in practice, consists of several sequential stages, the investor enjoys a relative freedom to change managers in between separate stages. Such option is readily exercised by the investor if she tends to think (attach some small probability) that the manager is biased toward particular decisions what is, in general, unfavorable for the investor. At the same time the manager desires to secure an opportunity to perform the whole project by himself and so he cares of the investor's opinion. Thus it appears rational for the manager to pretend to be an honest and professional one. But this can be done only by performing those actions which are interpreted by the investor to be unusual for biased manager. What in turn may force the manager to sacrifice expertise in favor of maintaining a good reputation. At the same time without reputational concerns just the presence of a sufficiently small fraction of biased managers do not dramatically and discontinuously worsens the outcome. On the contrary just a bit of incomplete information on the side of investor affects the behavior of *all* managers, not only the biased ones.

Our model extends and generalizes those of Ely and Välimäki [11] and Morris [21]. Primary conclusion of these papers is that incentives of long-run players to build reputation in some cases may substantially lower their equilibrium payoffs. Rational desire of the agents to separate from bad types and avoid getting bad reputation forces other players to respond in a way that is initially unfavorable for these agents.

We extend the analysis given in these papers in a number of ways. First my model allows to considers principal-agent framework with one long-run principal (investor) and a population of long-run agents (managers), while in the basic model of Ely and Välimäki there were many short-lived principals and one long-lived agent. Such extension makes it possible to explore the *sequential investment* framework. As it is widely recognized in the literature [26], such framework with greater degree of precision captures the relevant *dynamic* features of real world investment projects than relatively simple static models of investment process. It is intuitively appealing to think about such projects as consisting of several but finite number of stages. So we also consider a finite game without discounting in contrast to infinite horizon game in their model. Also the assumption about the population of long-run agents captures the real-life availability of many professionals that are readily hired in the market – it is evident that companies often hire new managers developing more attractive remuneration schemes.

The second substantial difference of our paper is that we are approaching a question of existence of a mechanism that will mitigate the bad reputation effects. This problem was not analyzed in any of the previous papers that considered bad reputation. In this respect we also extend the literature on repeated principal-agent models and in particular on optimal organization in a sequential investment setting. Our result that inferior outcomes may be caused by reputational concerns of the agents permits to view the problem of adverse selection in principal-agent framework under completely new perspective. It appears that in the presence of asymmetric information the agents will require additional compensation of particular structure in order to eliminate there incentives to "build" reputation. Intriguing conclusion of our analysis is that agents need to be paid, basically, for refraining from caring of their reputation.

One more distinction of our model is that we consider a reputational effects that arise among one principal and one agent – that is is we consider reputation as a private belief of what an agent is, rather than public one. Such structure is more realistic when, for example, the public record of relations of particular manager with other investors is not perfectly observed or is very costly to obtain. Then an investor left with the only option to form a beliefs about particular manager only on the grounds of there private interactions. Similar approach was undertaken by Morris [21], but he basically analyzed cheap talk game with completely different set of assumptions, while in this paper we consider principal-agent framework.

Interesting is that even when almost all agents are good in the sense that they are ready to make optimal investment decisions and it is costless for them the inefficiencies arise even in the presence of insignificantly small but still positive prior probability of the agent being of a biased or bad type. A tiny possibility of dishonesty dramatically changes equilibrium outcome and induce the necessity for specific compensation mechanism.

2.2 Literature Review

A usual approach to modeling a reputation is to incorporate "right kind" of uncertainty into a model. It was first done in the seminal papers of Kreps and Wilson [17], Milgrom and Roberts [22], Kreps, Milgrom, Roberts and Wilson [16]. First two papers proposed similar solutions to the so-called "chain-store paradox" described by Reinhard Selten in [29]. The paradox essentially exhibited the limits of backward induction in finding a perfect equilibrium of a finite stage game. The subgame perfect equilibrium of the game in which chain-store accommodated every entrant was initially practically doubtful and intuitively unappealing. The idea explored in these two papers was to consider an incomplete information game where the payoff function of the chain-store is not observed by other players. It appeared that introduction of such uncertainty results in an equilibrium where, roughly speaking, chain store maintains a reputation of being tough by fighting every entrant till some number with probability one and, as a result, no one from those entrants actually enters. So the small perturbation of information structure of the initial game led to qualitatively different and more realistic result. What is also important is that only the *beliefs* of the agents and not the "physical" possibility of particular type of chain store matters.

Similar approach permitted a cooperation to emerge in finitely repeated prisoners dilemma [16]. Thus explaining why the players were able to improve upon a pareto-inefficient outcome of complete information game.

Notable observation here was that uncertainty over the types of players gives them an opportunity to pretend to be someone else and so to extend a set of feasible equilibrium payoffs over that of complete information game even when horizon is finite. Such ideas were generalized later in Fudenberg and Levine [13], [14]. They first explored and characterized the bounds player's payoff in such games. There primary result was that long-run player is able to receive almost his Stackelberg payoff in a game perturbed by introduction of a small amount of incomplete information. Where Stackelberg type of a player is the player committed to only one pure strategy (Stackelberg strategy). And Stackelberg payoff is an equilibrium payoff of this player in a complete information version of the game. Intuitively this means that incomplete information in fact allows the player to commit himself to certain strategy what is otherwise impossible. Such commitment in an obvious way widens the possibilities of the player and accordingly leaves him better-off. This is a general conclusion of a considerable bulk of literature on reputation.

Morris [21] presents two-period cheap talk game where an informed advisor tries to convey her socially valuable information to a decision maker who decides what decision to implement. Beside the fact that the preferences of the advisor are identical to those of the decision maker, if the last one believes there is a positive probability that an advisor is biased toward a particular action, all the socially valuable information can be lost in the first period. The reason for this is following: if the advisor cares a lot of the outcome of a second period she may want to signal her type recommending the affirmative action even if she obtained a different signal, thus separating from the bad type. The decision maker knows this and so ignores the adviser in the first period. Thus the only possible equilibrium is with a "babbling" in the first period. Such theory, as the author points out, can be used to explain particular aspect of "political correctness". It is clear from this model that reputational concerns of the advisor can hurt efficiency and moreover are responsible for lowering advisor's payoff.

Ely and Välimäki [11] elaborate on the model of Morris considering the setup with a sequence of short-run principals and one long-run agent. In their example a mechanic chooses what repair to perform while a motorist decides to bring or not his car to the mechanic. The mechanic as an expert knows the right repair but he also is aware that the motorist suspects him to be the one who always prefers a particular single type of the repair independently of the realized state. Thus a sufficiently patient good mechanic may prefer to separate from bad type performing the other kind of repair. This in turn keeps infinitely inpatient motorists away – and so the market collapses. Also bad reputation result of Ely and Välimäki holds with rational (not a commitment) bad type.

Authors identify two strategic themes in a given model: inability of the mechanic to commit not to invest in reputation and inability of short-run motorists to internalize arising information externality. They also show that the last problem can be solved if the motorist is also a long-run player. Than, as it is shown in the paper, he can achieve an average payoff close to the full-information value (the case without incomplete information) even when the discount factor is approaching one for both players. This result is different from our's – in our model beside that all players are long-run inefficiency still remains. We later show that such conclusion is driven by the introduction of population of the agents with constant priors. Significant difference of the paper by Ely and Välimäki from the paper by Morris is that in the first one, performing wrong actions is costly for the agent, while in the second paper an advisor sends messages for free.

Examples of Morris and of Ely and Välimäki have a lot in common. In both papers reputational concerns of the players worsen equilibrium outcomes of the game in general and in particular decrease there equilibrium payoffs. This is in contrast with usual approach to considering reputation as something good for the long-run player. Natural question to ask is – what are the general properties of the game that lead to such bad reputation results? Ely, Fudenberg and Levine [10] generalize the notion of bad reputation game, identifying the necessary and sufficient conditions under which a given game is a bad reputation game, but they still consider the framework with one long-run and many short run players.

Model of sequential investment project developed in my paper is quite realistic in the sense that it captures the dynamic aspect of implementation investments. Sequential investments *per se* are discussed in detail in [26]. Among the most relevant examples of such investments are R&D projects, natural resource development etc.

Nevertheless a problem of optimal organization of sequential investment project is quite novel. Tamadaa and Tsai [30] discuss the issues of integration and separation problem in a two-stage model. That is they discuss the conditions under which it is better to assign one agent to both stages or two separate agents for each. Laux [19] investigates the problem of optimal contracting in repeated principal-agent model with multiple projects. So far there is no literature concentrating on the role of reputational incentives in designing optimal contracts under the sequential investment.

2.3 Model Setup

There are two-stage sequential investment projects each described by a vector $\mathbf{w} \in \Omega^2$, where $\Omega = \{h, l\}$ and $\omega_i \in \Omega$ is an amount of investment needed (high or low) for an optimal completion of stage $i = \overline{1,2}$ of a project. ω_i is an i.i.d. random variable with $Prob\{w_i = h\} = \gamma > 0$ and $Prob\{w_i = l\} = 1 - \gamma$, $\forall i$, so γ is a probability that completion of a particular stage requires high level of investments. The distribution of ω_i induce the distribution of $\mathbf{w}$.

Consider following Bayesian game Γ_2. There are 3 players – one long-lived investor and a pool of 2 long-lived managers. The investor faces an investment opportunity but can only roughly evaluate its type (investor is not an expert in a particular field) – she is aware only of the probability distribution over possible types, induced by observed signal $\gamma \in (0,1)$. She can for example infer probability distribution from past experience with similar projects in a given industry. The assumption that the investor observes a signal only in the beginning of the game and not at the beginning of each stage resembles the idea that she can only learn some *general* feature of the project that does not change with the implementation of more stages, while at the same time being ignorant about particular aspects of implementation. Nevertheless this aspect is of minor importance to subsequent analysis. Then the investor decides whether to start a project or to abandon it. If the investor starts the project she cannot implement it by herself and so, needs to hire a manager from a pool of homogeneous ones. The investor is not informed about the quality of any particular manager from the pool but she knows some constant characteristics of a typical manager. Then the two-stage game begins.

The hired manager is a professional who possesses an expertise. At first stage the manager perfectly observes needed amount of investment, that is he observes private signal ω_1 (he learns whether high or low investments are needed). Note that the manager gets an information only on current stage and is ignorant of all subsequent stages before he actually starts working on each of them. The manager then decides on his action $a_1 \in \Omega$ – he can either spend needed amount of money on investments (that is making use of available private information) or to arbitrary pick this amount.

When low investments are made when high where needed then it is clear that the probability of success must go down. Otherwise when low investments are needed and high investments are made it is reasonable to assume that probability does not changes at all – there may be for example some physical constraints that keep the success uncertain even under high investments. Anyhow this assumption is not crucial to our primary result. Formally, after the manager takes his action, stage either succeeds with probability:

$$Prob\{\text{Success of } i|a_i\} \equiv \begin{cases} \eta, & \text{if } \omega_i = a_i \ \vee \ \omega_i = l \\ \delta, & \text{if } \omega_i \neq a_i \ \wedge \ \omega_i = h \end{cases}$$

where $1 \geq \eta > \delta > 0$. Or fails with complement probability $Prob\{\text{Failure of } i|a_i\} = 1 - Prob\{\text{Success of } i|a_i\}$. If stage succeeds game proceeds to the next stage. If not – game stops and payoffs are realized. At the next stage the investor observes a public signal on amount of investments made by the manager at previous stage and decides whether to hire a new manager or to rehire the old one. Investor does not know what kind of investments were needed for completing preceding stages. Note that the only public signals that Investor can base the hiring decisions are the success or failure of a stage and the action of a manager. Then manager moves as described previously. Game lasts 2 stages. Payoffs are revealed at the end of the game.

The investor cannot commit herself not to fire manager. That is the investor or cannot sign an implementable contract with one manager for several stages or the costs of breaking such contract are negligible.

Manager can be of type $\theta \in \Theta = \{good, bad\}$ with $Prob\{bad\} = \mu$. It is a *prior* probability of a bad type in the sense that every manager drawn from a common pool is characterized by this probability before performing any actions.

Good type gets a fixed wage for the stage if hired and does not get wage if not hired. So his per stage payoff is:

$$\pi_i^g = \begin{cases} x_i & \text{if hired} \\ 0 & \text{if not hired} \end{cases}$$

Where $x_i > 0$ is the wage that the investor pays to the manager for completion of stage i.

The bad type experience additional benefits of performing expensive investments. ϵ_i is an amount of such benefits that can be obtained in addition to wage after playing h at stage i. So bad type has h as a weakly dominant stage strategy independently of the realized value of ω:

	h	l
Hire	$x_i + \epsilon_i$	x_i
Not hire	0	0

Figure 2: Payoff of a rational *bad* type per stage i

where $\epsilon_i > 0, \ \forall i$. Assume for a moment that the hired manager gets fixed wage independently of success or failure of the stage. The manager maximizes payoff of a whole game $\Pi = \sum_{i=1}^{2} \pi_i$.

To simplify notation and analysis we can without loss of generality focus further only on pure strategies of the players.

An important feature of sequential investment projects is that success of the whole project depends on the success of each particular stage. In particular we assume that for the success of the project every stage must have succeed. For example when pharmaceutical company develops a new drug it is engaged in several stages of research and testing. If any stage fails, maybe because of physical impossibility of developing particular drug, the project also fails and the company does not have to implement all subsequent stages. The strategy profile of the game together with realized type of the project induces probability distribution over the success of each stage. All these implies a particular structure on the payoff of the investor:

$$U = \begin{cases} V(\mathbf{w}) - C_1 - C_2 & \text{if each stage succeeds} \\ -C_1 & \text{if first stage fails} \\ -C_1 - C_2 & \text{if second stage fails} \end{cases} \tag{4}$$

Here $V(\mathbf{w}) \in \mathbb{R}_{++}$ is an expected payoff of a project of type $\mathbf{w} \in \Omega^2$. At this point we do not make particular assumptions about $V(\mathbf{w})$. Costs of investments at stage i are

$$C_i = \begin{cases} H, & \text{if } a_i = h \\ L, & \text{if } a_i = l \end{cases}$$

where $H > L > 0$.

Note that the investor invests only if previous stage was successful.

The investor decide first whether to start or not the project (and hire manager for the first stage) and than whether to hire a new or rehire the old one. So the set of pure actions of the investor is $S = \{start, not\ start\}$ in the beginning of the game and $S = \{new, old\}$ after the successful completion of the first stage.

The investor has an outside option available that gives the reservation payoff $\bar{U} > 0$. So she invests in project *if and only if* $EU \geq \bar{U}$. Such outside option may be interpreted as a possibility to invest in risk free asset and get fixed rate of return.

Assumption 1 *Expected utility of a project of type $\boldsymbol{w} \in \Omega^2$ is maximized when $a_i = \omega_i,\ \forall i = 1, 2$. That is:*

$$\boldsymbol{w} = \arg\max_{\boldsymbol{a} \in A^2} EU$$

This assumption says that the project gives maximal expected payoff if it is completed *optimally*. To see that it is not evident form the beginning recall that C_i depends on a_i. This is basically the reason for a phrase "optimal completion of the project" in the first passage.

2.4 Model Solution

To simplify analysis suppose that $\eta = 1,\ \delta > 0$. We are to find a perfect Bayesian equilibrium of this game.

So first lets define a perfect Bayesian equlibrium (PBE).

Definition PBE of Γ_2 is a strategy profile and corresponding beliefs that satisfy:

1. strategies are *sequentially rational*
2. $Prob\{bad\} = \mu$
3. $Prob\{bad|h\} = \bar{\mu}(h) = \frac{Prob\{bad \bigcap h\}}{Prob\{h\}}$

As far as there are no any intrinsic reasons for good manager to discriminate between high and low investments it is interesting to consider further only cases when he chooses the right amount while being indifferent.

It is apparent that when there is no bad type ($\mu = 0$) there will be an equilibrium where the manager if hired always invests needed amount and projects are always completed optimally. That is the investor expects highest possible payoff for a given γ. This result is in the sharp contrast to the results when $\mu > 0$.

Consider the system of beliefs:

$$\bar{\mu}(h) = 1,\ \bar{\mu}(l) = 0$$

The strategy of the investor:

$$\sigma(start|\ \emptyset,\ \gamma \in \Psi(\bar{U})) = 1,\ \sigma(start|\ \emptyset,\ \gamma \in \mathbb{R} - \Psi(\bar{U})) = 0$$

$$\sigma(new|h) = 1,\ \sigma(new|l) = 0$$

The strategy of the manager:

$$\begin{aligned} &(0,0), \forall\ \theta \in \Theta\ \wedge\ i = 1 \\ &(1,0)\ \text{ if } \theta = good\ \wedge\ i = 2 \\ &(1,1), \text{ if } \theta = bad\ \wedge\ i = 2 \end{aligned}$$

Proposition 3 *Suppose $\epsilon_1 < (x_2 + \epsilon_2)\delta$. Then beliefs and strategy profile given by above constitute a PBE of* Γ_N[6]

That is if the additional benefit of a bad type from choosing high level of investments at the first stage is less than the expectation of his payoff from choosing high investments in the second period under the probability of success equal δ then every type will chose low expenditures in the first period independently of realized state. So it follows that the bad type will pretend to be the good one always choosing low level of investments and the good type will be forced to follow the similar strategy.

Proof Strategies of the manager for $i = 2$ are immediately derived using backward induction.

It is always sequentially rational for the investor to hire a new manager after the first stage if $\bar{\mu} > \mu$ as far as expectation of continuation payoff for a second period will be unambiguously lower $\forall \gamma \in (0, 1)$ if the old manager is rehired.

Thus the investor will play after the first stage a pure strategy *hire new* if $\bar{\mu} > \mu$, and pure strategy *rehire old* if $\bar{\mu} < \mu$. Suppose she always plays *rehire old* if $\bar{\mu} = \mu$.

By Bayes law given mentioned strategies obtain:

$$\bar{\mu} = \frac{Prob\{bad \bigcap h\}}{Prob\{h\}} = \mu$$

Consider $\theta = good$. Whatever is observed ω_1 playing h with positive probability may result in loosing x at second stage and thus is not optimal while l secures the possibility of being rehired.

If $\theta = bad$ playing h with positive probability results in getting ϵ_1 immediately but loosing $(x_2 + \epsilon_2)$ at the second stage. Playing l manager secures at least $(x_2 + \epsilon_2)\delta$ in exchange for ϵ_1. Thus pure strategy given by (4),(6)is sequentially rational.

Then expected payoff to the investor is given by:

$$EU(\gamma) = \gamma^2 E\widetilde{V}(h, h) + \gamma(1 - \gamma)(E\widetilde{V}(h, l) + E\widetilde{V}(l, h)) + (1 - \gamma)^2 E\widetilde{V}(l, l)$$

[6]There are basically two more relevant equilibria (depending on values of parameters) with strategies of bad type in the first period given by (1,0),(1,1) and of a good type by (0,0).

Where

$$E\widetilde{V}(h,h) = V(h,h)\delta - L - H\delta$$

$$E\widetilde{V}(h,l) = V(h,l)\delta - L - L\delta$$

$$E\widetilde{V}(l,h) = V(l,h) - L - H$$

$$E\widetilde{V}(l,l) = V(l,l) - L - L$$

Expected payoff from first two projects (h,h) and (h,l) is lower comparing to optimal completion. In general expected payoff is not monotone with respect to γ. Define

$$\Psi(\bar{U}) = \{x \in \mathbb{R} : EU(\gamma) \geq \bar{U}\}$$

These are the values of γ that give the investor positive expected payoff. Then $\gamma \in \Psi(\bar{U})$ is a participation constraint for the investor. *Q.E.D*

In a given equilibrium low level of investments is always chosen by the managers at the first stage. Good manager tries to separate from bad one but bad also wants to secure the possibility of being rehired for the second stage and then play h. Primary observation here is that expected payoff to the investor higher for all γ under zero probability of bad type than under positive one. This point is illustrated in figure 2 for some particular values of parameters (here $V(\mathbf{w})$ is assumed to be constant). This suggests that for high enough $\bar{U}$ less projects will be implemented. Moreover the result is independent of prior $\mu > 0$: even for arbitrary small positive values of μ incentives to separate on behalf of good manager will induce inferior equilibrium in the sense that not all initially attractive projects are implemented. The managers independently of type and signal received make low investment in the first period. This leads to lower expected payoff for projects that have greater probability of need for high investments, that is γ, in first period, all other things constant.

Interesting interpretation of the result is as an example of the cost of dishonesty like in celebrated Akerlofs paper [1]. Existence (or even belief in existence) of small fraction of dishonest managers in the presence of information asymmetries discontinuously distorts market efficiency crowding out initially desirable investment projects. Driving force behind such process is the incentives of good managers to separate and the ability of the investor at no cost to fire and hire new managers from a pool with constant priors. Moreover it is worth noticing that the assumption of the existence of such

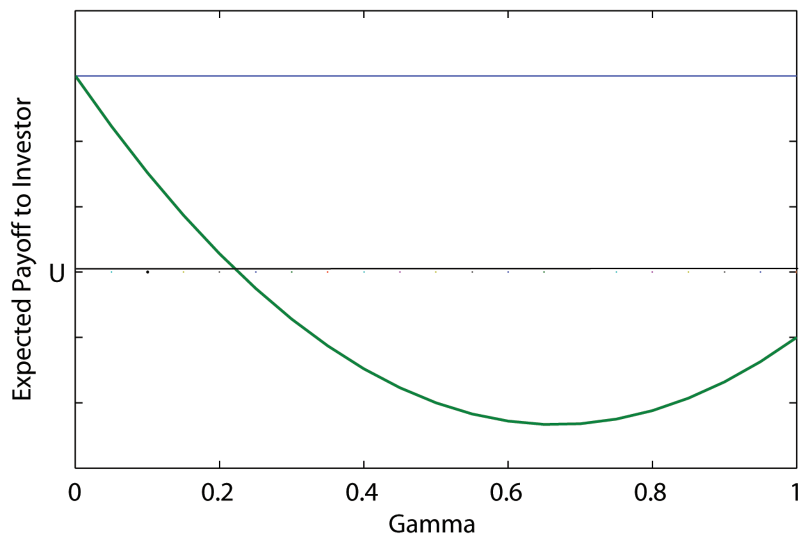

Figure 3: Expected payoff as a function of $\gamma \in (0, 1)$

a pool of agents, beside being quite natural[7] is a source of substantial discontinuity in my model. This assumption restricts conditional probability of the manager being of a bad type to be always not greater than the prior, what in turns compromise the option of sending bad signal by a good type, thus leading to an equilibrium where everyone ignores private signal in initial period.

2.5 Optimal Wage Schedule in a Two Stage Model

It was shown in a previous section how reputational concerns of the managers may lead to decrease in expected payoff to the investor and inefficient investment decisions. So the natural question that arises is can be such a problem solved using some specific incentive inducing structure. The investor may want to design a wage schedule for the manager that somehow will decrease incentives to build reputation thus insuring better outcome for the investor. To consider such a problem we need additional assumptions.

Let $\bar{x}$ is a reservation payoff of the manager for each stage. The manager

[7]It is quite realistic that there are many homogeneous, from the view of Investor, managers that are readily hired.

agrees to participate if only if expectation of stage payoff is greater than $\bar{x}$. Such assumption may seem strange – really it is plausible to think that the manager will agree to get little wage for one period if he will be compensated in other period thus keeping average wage above $\bar{x}$. But at least in two stage game it is impossible in sequential equilibrium with the availability of a pool of homogeneous agents.

In previous section we assumed that the manager gets the wage whatever the outcome of a stage is. Now we drop this assumption. The investor can condition wage and hiring decisions only on observed public signals – actions of the manager in the previous stages and success or failure of previous stage. So the optimal wage schedule must condition wage on public signals and induce the highest possible equilibrium expected payoff to the Investor. Note that it was assumed that the wages do not enter explicitly to payoff the investor. This assumption can be easily relaxed but it can be also motivated by the fact that the amount paid in the wages is insignificant comparing to expected costs and benefits of a project alone. But we still are interested in the *optimal* wage schedule in the sense that it is the less costly among those inducing highest possible equilibrium payoff.

For simplicity we make following assumption. Later it can be relaxed.

Assumption 2 *The wage paid to a manager depends only on the outcome of a particular stage.*

We will also impose limited liability constraint on the wages paid to a manager: $x_i \geq 0, \ \forall i$.

Definition A wage schedule is a vector $\langle (x_1^s, x_1^f), (x_2^s, x_2^f) \rangle \in \mathbb{R}_+^4$ where x_i^s is a wage if stage succeeds, x_i^f is a wage if stage fails.

Before defining optimal wage schedule notice following fact.

Claim 4 *There is no wage schedule* $\mathbf{x} \in \mathbb{R}_+^4$ *such that bad type plays "low" at the second stage.*

Proof Claim trivially holds as far as playing h never lowers the probability of success, second stage is a terminal one and h is a dominant strategy for a bad type.

Q.E.D

Now lets define optimal wage schedule for a particular two stage case more precisely:

Definition Vector $\mathbf{x} \in \mathbb{R}_+^4$ is an *optimal wage schedule* if it induces the existence of equilibrium profile with strategy of manager given by:

$$\langle (1,0)\ \forall\theta\ \wedge\ i=1;\ (1,0) \text{ if } \theta = good\ \wedge\ i=2;\ (1,1) \text{ if } \theta = bad\ \wedge\ i=2\rangle \quad (5)$$

That is the manager always plays $a_i = \omega_i$ beside the bad type at the second stage, who plays h in any case.

Such definition is very strict. Actually if μ is sufficiently small the investor will care only about the behavior of good type. So the problem can be reduced to finding a wage schedule that induces good type to perform the right action in the first period and is the cheapest among all such schedules. This task is straightforward.

Proposition 5 *Let* $\Lambda \equiv \frac{\eta-\delta}{\eta\delta}$ *and* $\frac{\bar{x}}{\eta} = \hat{x}$. *The wage schedule that induces PBE with good type strategy given by* $(1,0)$ *in every period is:*

$$\langle (x_1^s, x_1^f), (x_2^s, x_2^f)\rangle = \begin{cases} (\hat{x}, 0, \hat{x}, 0) & \text{if } \Lambda \geq 1 \\ \left(\frac{\hat{x}}{\Lambda}, 0, \hat{x}, 0\right) & \text{if } \Lambda < 1 \end{cases}$$

The proof is evident from the subsequent discussion.

Note that $\frac{\partial\Lambda}{\partial\delta} < 0$ from what follows that total payment to the manager decreases with δ till it equals $\hat{x}$. This result is quite intuitive because the greater the risk that project will fail when low investment instead of high are maid, the greater the expected costs of separating for a good manager. Thus it will be cheaper to suppress his reputational incentives.

Given wage schedule is implementable in the sense that it does not faces the problem of renegotiation. We call this property renegotiation stability. The second period wage is the minimal possible and so it could not be changed anyway.

Optimal wage schedule appears to be quite complicated.

Suppose strategy of Investor is $\sigma(new|h) = 1,\ \sigma(new|l) = 0$.

Consider incentive compatibility constraint for a good type. If $\omega_1 = h$ then $a_1 = h$ if and only if

$$x_1 > x_1\delta + x_2\eta\delta$$

If $\omega_1 = l$ then $a_1 = l$ if and only if

$$x_1\eta + x_2\eta^2 > x_1\eta$$

Second constrain does not bind. It follows that strategy $(1, 0)$ at first stage is optimal for a good type if:

$$\frac{x_2}{x_1} < \frac{\eta - \delta}{\eta\delta} \equiv \Lambda$$

Now consider incentive compatibility constraint for a bad type. If $\omega_1 = h$ then $a_1 = h$ if and only if

$$(x_1 + \epsilon_1)\eta > (x_2 + \epsilon_2)\delta\eta + x_1\delta$$

If $\omega_1 = l$ then $a_1 = l$ if and only if

$$(x_1 + \epsilon_1)\eta < (x_2 + \epsilon_2)\eta^2 + x_1\eta$$

From what follows

$$(x_2 + \epsilon_2)\delta\eta + x_1\delta < (x_1 + \epsilon_1)\eta < (x_2 + \epsilon_2)\eta^2 + x_1\eta$$

Assume that $x_i^f = 0$.

To find *optimal* wage schedule solve linear minimization program with constraints given above:

$$\min_{x_1, x_2} \{x_1 + x_2\eta\}$$

subject to

$$\frac{x_2}{x_1} \leq \Lambda$$

$$(x_2 + \epsilon_2)\delta\eta + x_1\delta < (x_1 + \epsilon_1)\eta < (x_2 + \epsilon_2)\eta^2 + x_1\eta$$

$$x_1 \geq \hat{x}$$

$$x_2 \geq \hat{x}$$

This problem appears to have a solution where x_2^s may be necessarily greater than $\hat{x}$. But then it will be sequentially rational for the investor to try to renegotiate wage for second stage when the first is completed. As far as there are always managers in the population who will be willing to work only for $\hat{x}$ such renegotiation is always possible. So if the investor cannot commit himself to a given mechanism it may be impossible to induce the equilibrium profile where the bad type chooses the optimal action at the first stage. That is the optimal wage schedule that is renegotiation stable need not necessarily exist.

2.6 Conclusions

Dynamic nature of sequential investment projects coupled with the incomplete information about the incentives of hired managers drives the dramatically inefficient investment decisions. The result holds even when the probability of a manager to be of a “bad”, that is biased, type goes to zero. The main reason behind such observation is that not the direct influence of bad type matters but how it’s presence indirectly affects the decisions of good managers. Incentives of these managers to avoid “bad” reputation separating from bad type hurt efficiency and cause the drop of the expected return on investment projects. This naturally confirms the stylized fact that more expensive investments project are harder to implement.

Moreover the wage schedule that partially mitigates the given problem is explored. It is shown the schedule that induces good type to make optimal investment decisions is renegotiation stable. On the contrary it is show that in general there is no renegotiation stable wage schedule to induce the optimal decision by the strategic bad type at least at the first stage of the investment project.

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