PERFECT BAYESIAN EQUILIBRIA IN A CLASS OF TWO-STAGE AUCTIONS

by

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Abstract

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This research investigates perfect Bayesian equilibria of a two-stage ProZorro-type auction with two asymmetric bidders. We examine the possibility of interpreting the second stage of the auction as the resale stage of the corresponding asymmetric first-price sealed-bid auction with resale. It is shown that the equilibrium exists and is unique. Furthermore, the equilibrium strategies in the asymmetric two-stage auction differ from the equilibrium strategies in the asymmetric first-price sealed-bid auction with resale. In addition, contrasting to the auction model with resale, the second stage of the ProZorro-type auction does not symmetrize the auction with initially asymmetric bidders.

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Chapter 1

INTRODUCTION

The classical auction theory is based on the two types of auctions, which are the first-price sealed-bid auction and the second-price sealed-bid auction. The first-price sealed-bid auction describes the situation when the bidder with the highest bid submitted during the simultaneous bidding process wins the object and pays her own bid to the auctioneer. On the other hand, the second-price sealed-bid auction works under the rule, according to which the owner of the highest bid wins the object, but pays the second-highest bid to the auctioneer. Despite the fact that these auctions are sealed-bid, there are corresponding open auctions, which reflect the same properties and are strategically equivalent to the described above. These are English open ascending price auction and Dutch open descending price auction. As defined by Krishna (2010) the Dutch auction provides the information that some bidder agreed to buy the object at current price, however, this leads to the end of the auction. Every bidding strategy in the Dutch auction has its equivalent strategy in the first-price sealed-bid auction and vice versa. Rather similar, but weaker relation has the English open ascending auction and the second-price sealed-bid auction. Albeit in the English auction it is the most preferred strategy to bid the value, likewise in the second-price sealed-bid auction, these two auctions are not strategically equivalent. Moreover, the optimal strategies in these auctions coincide only under the assumption of independent private values. Hence, there is a strong equivalence between the Dutch open auction and the first-price sealed-bid auction, and a weak equivalence between the English auction and the second-price sealed-bid auction.

The first-price sealed-bid auction and the second-price sealed bid auction are themselves equivalent in terms of the revenue brought to the auctioneer and the efficiency but under particular assumptions of independent private values, risk neutrality, and symmetry among bidders. Once any of these assumptions is released, the revenue equivalence vanishes. Furthermore, the asymmetry in distributions of bidders' private values results in the inefficient allocation in the first-price sealed-bid auction. The possibility of increasing the efficiency of the auction and the revenue of the auctioneer, as the two major characteristics of any auction, is sufficiently studied. For instance, Hafalir and Krishna (2008) considered the asymmetric first-price sealed-bid auction with resale.

The model investigated represents a classical first-price auction with two bidders with asymmetrically distributed private values. However, once the winner is defined and the auction is over the resale takes place. The winner inferencing that the bidder with losing bid has a higher valuation of the object, which means the auction was inefficient, makes "take-it-or-leave it" offer to the loser. If she accepts, the resale occurs via monopoly pricing. Hafalir and Krishna obtain the equilibrium and prove that it is unique, they define that resale possibility increases the expected revenue of the auctioneer from the first-price sealed-bid auction, and it surpasses the expected revenue from the second-price sealed-bid auction. However, since the post-auction resale occurs under incomplete information the allocation of the object is not always efficient.

This thesis follows the work of Hafalir and Krishna (2008) in considering the possibility of increasing the revenue of the auctioneer from the first-price sealed-bid auction, however, using the concept of multi-stage sequential auction. The resale stage could be considered in itself as the second stage of the two-stage auction with one difference that the object changes its owner after resale took place, while in the two-stage auction property rights on the object remain in the hands of the auctioneer. Hence, the main issue to investigate is whether the equilibrium of the first-price sealed-bid auction with resale could be applicable to the two-stage auction under the same assumptions.

The multi-stage auctions are a prosperous basis for research, since they are mostly used in public procurements. One of the bright examples of such auctions is a model applied in Ukraine for conducting public procurement, an electronic system based auction "ProZorro". This auction consists of three stages of sequential bidding with zero-round of the simultaneous bidding process. The efficiency of "ProZorro" could be estimated now only from the empirical evidence, since the theoretical approval of multi-stage auction, particularly the model used in "ProZorro", has not been sufficiently studied. Moreover, even the cost reduction for the government due to using of "ProZorro" is proved empirically, the question whether any simpler model, e.g. one-stage first-price auction, could be more efficient than the model used remains somewhat controversial.

The main method of determining the efficiency of any public procurement is the revenue brought to the auctioneer, i.e. government. However, in order to calculate the revenue of the auctioneer, there should exist the equilibrium state, which is described by the absence of incentives to deviate for the players. This equilibrium reflects the optimal position for all players and includes optimal strategies, which bidders follow. Thus, the thesis has multiple objectives to be achieved.

Firstly, we are to investigate asymmetric two-stage auction for the existence of any equilibrium and define whether it is unique. Secondly, we are to check the possibility of implementation the equilibrium strategies from the asymmetric first-price sealed-bid auction with resale described by Hafalir and Krishna (2008) into the two-stage auctions with two bidders under the assumption of asymmetrical bidders. Consequently, we are to define whether the resale possibility is identical to the second stage in terms of the equilibrium strategies of the bidders and whether the property rights affect bidding behavior.

The thesis is organized in the following way. Chapter 2 overviews the main literature on the first-price auctions with extensions. Chapter 3 describes the methodology of determining perfect Bayesian equilibria. The main findings of Hafalir and Krishna (2008) in determining equilibrium strategies in the first-price sealed-bid auction with resale are revised in Chapter 4. The main results of the asymmetric two-stage auction investigation are presented in Chapter 5. Chapter 6 concludes main findings and provides policy implication of the thesis.

Chapter 2

LITERATURE REVIEW

This chapter describes the basic papers concentrated on the first-price sealed-bid auctions with extensions and under different assumptions and provides theoretical approval of tools chosen for the thesis.

The majority of auction models studied do not consider the possibility of further resale, however, it could have a considerable impact on players' equilibrium strategies and consequently on the auctioneer revenue, which is the question of interest in researches conducted in the auction theory. Moreover, the introduction of resale possibility approaches the one-stage auction to the two-stage auction, however, it distorts the behavior of buyers due to the change of property rights after the first round. In addition, asymmetry among bidders in the first-price auction leads to inefficient allocation, but the existence of resale condition might solve such a problem. Hafalir and Krishna (2008) investigate this question in the basic paper for our research. They consider "the post auction resale under monopoly pricing, when the winner of the auction makes a "take-it-or-leave-it" offer to the loser". The results obtained are the following: despite the initial asymmetry assumption, the distributions of bids of the two bidders are revealed to be identical in equilibrium; asymmetry has also some impact in the process of post-auction resale. Should resale be conducted under the imperfect information, the allocation would not be always efficient; the possibility of resale also changes the equilibrium in the second-price sealed-bid auction and it fails to be optimal for the player to bid her own value. The main conclusion authors come to is that once resale possibilities are admitted, the expected revenue from a first-price auction exceeds that from a second-price auction, but only in the case of monopoly pricing in the process of post-auction resale.

Similar research is also conducted by Hafalir and Krishna (2009) when they compare the equilibria in the first-price auction without resale with the first-price auction when resale possibility is introduced. They study the possible impacts of the resale possibility on the efficiency of the first-price auction and the expected revenue of the auctioneer obtained in the process of auction with such an extension. Moreover, they also consider the possibility of post-auction resale affecting the behavior of bidders while the auction takes place. The main result is the positive effect of resale possibility on the revenue of the original seller, but it might lead to some decrease in efficiency of the object allocation.

There are a lot of published papers concentrated on the auction theory; however, almost all the authors make use of the same tools for constructing the model, determining primitives of the model, such as core assumptions, equilibrium bidding strategies, revenue of the auctioneer etc. Vijay Krishna (2010) provides the theoretical basis for the research with sophisticated mathematical tool needed to describe the auction model. The author describes the basis of the auction theory, determines the basic principles of classical first-price and second-price sealed-bid auctions with symmetric bidders. Moreover, Krishna also considers extensions to the classical models, describes the methodology of investigation the auction with asymmetric players for existence of equilibria and determines approaches of defining the expected revenue of the auctioneer, which are explicitly used in this thesis.

Any auction is aimed at simplifying the process of selling the object with contemporaneous increase in the final price, which seller obtains from the winner. So the main problem of the seller is to find such an auction procedure that allows for the highest expected revenue to be obtained. The problem of the auction design is described in one of the most classical papers written by Mayerson (1981). He considers the seller being in the conditions of imperfect information, since she does not know how much bidders want to buy the object and what amount each of these bidders is willing to pay for the object, which is their values. The seller problem is defined further almost in the same way as of any bidder's, because any bidder knows only her own value and has to make inference about the values of other bidders as just does the seller. Moreover, a seller and bidders are assumed to be risk neutral, which means they maximize their utility expressed in terms of payoffs. The main finding is the impossibility for existing a unique auction model guaranteeing the realization of the object together with bringing the highest expected revenue for the auctioneer under any circumstances.

There are two major estimates of the auction aimed to define the better one that is, as mentioned above, the expected revenue it brings to the seller, and the efficiency of the auction. Efficiency in this case is to be considered as allocation of the object in the hands of a bidder that values it the most. Since the asymmetry condition results in inefficient allocation via the first-price auction, Vijay Krishna (2002) chooses the ascending price auction to check the existence of such circumstances under which a well-known mechanism, English auction, allocates efficiently. He considers the equilibria in the n-bidder single-object English auction under the assumption of interdependently and asymmetrically distributed private values, since the interdependence of values in the ascending price auction proved to bring the higher revenue than both first-price and second-price sealed-bid auctions, but the efficiency of allocation, when asymmetry is introduced, remains an open question. Krishna extends a single-crossing condition stating that asymmetric English auction with interdependent values could be efficient as long as every bidder's signal has a greater influence on his own value than on any other bidder's value (Maskin, 1992). He introduces "average crossing" and "cyclical crossing" conditions. The former reflects "a single crossing condition between a particular bidder's value and the average of all bidders' values with respect to some other bidder's signal", when the latter requires the cyclical order of impact that different signals could conduct on different bidders' values. The main result of the paper is the fact that the n-bidder English auction could allocate efficiently under either of these "crossing" conditions.

The revenue equivalence principle holds for the first-price and second-price sealed-bid auction, but under particular assumptions such as symmetry of bidders, risk neutrality, and lack of collusion among buyers. E. Maskin and J. Riley (1998) study two asymmetric auctions, art auction and job-contract bidding. Since the former suffers from idiosyncrasy of players and different budget constraints, while in the latter there are asymmetric beliefs due to different possibilities of completing information gaps. Authors release the assumption of symmetry in order to determine whether the revenue equivalence holds by considering three different cases of asymmetry. As a result, no such equivalence is found to exist, when any asymmetry is introduced.

There are two mechanisms of buying/selling the object in considering the order of bidding, sequential and simultaneous auctions. For the purpose of determining the better one for both the seller and the buyer Fatima (2007) compares the outcomes of these auctions. They consider the multi object auction model with separate auction for each object with bidders interested in only one object. The auctions are conducted under English auction rules with the assumption of uniformly distributed private values. The result of the research shows that revenue of the seller is higher when sequential mechanism is used. However, seller's expected profit depends on the number of objects being sold via the auction and on the number of bidders participating, and the resulted expected profit is ambiguous with some cases higher for sequential auctions and otherwise for auctions with the simultaneous move.

Our contribution to the literature on asymmetric auctions is the extension of the model developed by Hafalir and Krishna (2008) in modifying the resale stage in asymmetric one-stage first-price sealed-bid auction into the second stage of asymmetric two-stage auction. We check for the applicability of the equilibrium obtained in the asymmetric first-price sealed-bid auction to the similar problem described via two-stage auction of simultaneous bidding process in the first stage and sequential bidding process in the second stage.

Chapter 3

METHODOLOGY

As defined above we are to compare the equilibrium strategies of the asymmetric first-price sealed-bid auction with resale with the asymmetric two-stage auction.

3.1. Sealed-bid auction

First, we should introduce a definition of a sealed-bid auction. Maschler, Solan and Zamir (2013) provide the following definition of a sealed-bid auction:

Definition: A sealed-bid auction (with independent private values) is a vector $(N, (V_i, F_i)_{i \in N}, p, C)$, where:

- $N = \{1, 2, \dots, n\}$ is the finite set of buyers.
- $V_i \subseteq R$ is the set of buyer $i's$ private values, for each $i \in N$.
- Private values V_i of each buyer in the set N are distributed according to the cumulative distribution function F_i.
- $p: [0, +\infty)^N \to \Delta(N)$ is a function, which associates each vector of bids $b \in [0, +\infty)^N$ with a distribution that identifies the winner of the auction.
- $C: N \times [0, +\infty)^N \to R^N$ is a function, which determines the payment each bidder pays, for every vector of bids $b \in [0, +\infty)^N$, depending on which buyer $i_* \in N$ is the winner.

3.2. First-price sealed-bid auction with asymmetric bidders

The basic model, on which the main two-stage auction model lies upon, is the classical first-price sealed-bid auction, but under the assumption of asymmetry among bidders. We use a calculation technique borrowed from Krishna (2010).

Suppose there are two risk-neutral bidders with independent private values X_1 and X_2 distributed according to the distribution functions F_1 on the support $[0, \omega_1]$ and F_2 on the support $[0, \omega_2]$, respectively. The corresponding densities are $f_1 \equiv F_1'$ and $f_2 \equiv F_2'$, which are assumed to be continuous and positive on $(0, \omega_1)$ and $(0, \omega_2)$, respectively. We also assume that both distribution functions F_i are regular, which according to Myerson (1981) stipulates that for $i = 1,2$ functions

$$x - \frac{1 - F_i(x)}{f_i(x)}$$

are strictly increasing.

Suppose that there exist equilibrium strategies of two bidders β_1 and β_2, respectively. These strategies are assumed to be continuous and strictly increasing, and have inverses ϕ_1 and ϕ_2 respectively, such that $\phi_1 = \beta_1^{-1}$ and $\phi_2 = \beta_2^{-1}$.

Since a bidder would not bid more than her value, $\beta_1(0) = 0 = \beta_2(0)$. Furthermore, $\beta_1(\omega_1) = \beta_2(\omega_2)$, because should it be the case that, for example, $\beta_1(\omega_1) < \beta_2(\omega_2)$, bidder 2 would win with probability 1 if her value is ω_2, and would have an incentive to bid slightly less than $\beta_2(\omega_2)$ in order to increase her payoff. Let $\beta_1(\omega_1) = \beta_2(\omega_2) \equiv \bar{b}$, where $\bar{b}$ is the common highest submitted bid.

Since the bidder $j = 1,2$ follows the strategy β_j, the bidder $i's$ $i \neq j$ expected payoff, if her value is equal to x_i and she bids $b < \bar{b}$, is

$$\Pi_i(b, x_i) = F_j(\phi_j(b))(x_i - b),$$

where $F_j(\phi_j(b))$ is the probability that $x_j \leq \phi_j(b)$ that is the winning condition, and the second term is the payoff of the bidder i. Hence, bidder i chooses such b that maximizes $\Pi_i(b, x_i)$.

The first-order condition with respect to b results in

$$0 = f_j\left(\phi_j(b)\right)\phi_j'(b)(x_i - b) - F_j\left(\phi_j(b)\right),$$

$$F_j\left(\phi_j(b)\right) = f_j\left(\phi_j(b)\right)\phi_j'(b)(x_i - b),$$

$$\frac{f_j\left(\phi_j(b)\right)\phi_j'(b)}{F_j\left(\phi_j(b)\right)} = \frac{1}{x_i - b}.$$

Since bidder $i's$ value is $x_i \equiv \phi_i(b)$, we can rearrange to the system of differential equations for $j = 1,2 \; and \; i \neq j$

$$\frac{d}{db} lnF_j\left(\phi_j(b)\right) = \frac{1}{\phi_i(b) - b}. \tag{1}$$

Solution to every equation of this system of differential equations reflects the equilibrium strategies corresponding to each particular bidder in the first-price sealed-bid auction with asymmetric bidders. However, an explicit solution is tedious to be found in general representation, but only in some particular cases for a certain distribution functions.

3.3. Perfect Bayesian equilibrium

The two-stage auction represents the dynamic game of incomplete information, the equilibrium in which could be determined by finding the perfect Bayesian equilibrium (PBE). Hence, we should introduce the definition of PBE. According to Gibbons (1992), PBE is defined as follows:

Definition: A perfect Bayesian equilibrium consists of strategies and beliefs satisfying the following requirements:

Requirement 1. The player, whose turn to move, must have a belief concerning the node reached by the game at any information set. For a nonsingleton information set, a belief is a probability distribution over the nodes in the information set; for a singleton information set, probability equaling to one must be put on the single decision node.

Requirement 2. The players' strategies must be sequentially rational according to given players' beliefs. It means that for any information set the player, whose turn to move, must make optimal decisions and have optimal following strategies according to the player's belief at this information set and the other players' following strategies (where a "following strategy" reflects a full set of actions, which might be taken after the information set reached).

Requirement 3. Should the information set lies on the equilibrium path, beliefs are to be defined by the means of Bayes' rule and resulting players' equilibrium strategies.

The difference between the information set lying on the equilibrium path and one lying off the equilibrium path, is determined as follows:

Definition: Should the players follow the equilibrium strategies, the information set will lie on the equilibrium path if probability for reaching this particular set is positive; and if the probability of reaching the set equals zero, which means the information set will not definitely be reached, this set lies off the equilibrium path.

Requirement 4. Should the information set lies off the equilibrium path, beliefs are to be defined by means of Bayes' rule and resulting players' equilibrium strategies where it is possible.

C h a p t e r 4

EQUILIBRIUM BIDDING STRATEGIES IN THE FIRST-PRICE SEALED-BID AUCTION WITH RESALE

The basic model taken for the description of the two-stage auction is the asymmetric first-price sealed-bid auction with resale investigated by Hafalir and Krishna (2008). Authors consider two risk neutral buyers with asymmetrically distributed independent private values participating in a classical first-price sealed-bid auction. After the end of the auction and announcing the winning bid a winner, if she wishes, might make a take-it-or-leave-it offer to a loser to buy the object. Should the loser accept the offer, the resale takes place; otherwise, the object remains a property of the winner. Since bidders are entitled to know ex-ante about the post-auction resale, they decide on bidding strategies taking into account the possibility of further resale.

The first-price sealed-bid auction with resale is a dynamic game of incomplete information, so the backward induction concept should be used when deriving the equilibrium strategies of the bidders. Hence, we first consider bidders' behavior in the resale.

Resale Stage

Suppose that there are two risk-neutral bidders with independent private values V_i, $i \in \{s, w\}$, where s is "strong" and w is "weak", distributed according to distribution function F_i with support $[0, a_i]$. The notion of "strong" and "weak" bidders correspond to the first-order stochastic dominance, that is for all v,

$F_s(v) \leq F_w(v)$, which means that distribution of values of the strong bidder stochastically dominates the corresponding distribution of the weak bidder. These bidders are assumed to follow bidding strategies β_s and β_w, which are continuous and strictly increasing and have inverses ϕ_s and ϕ_w, respectively.

Suppose that bidder j wins the auction with a bid equals to b, the value of bidder j is v_j. The resale will take place only when there are potential gains from trade, that is if $v_j < \phi_i(b)$. Also bidder j assumes that $V_i \leq \phi_i(b)$. In the resale stage bidder j has to choose such p, which solves

$$\max_p \left[F_i\big(\phi_i(b)\big) - F_i(p)\right]p + F_i(p)v_j, \tag{2}$$

The term in brackets reflects the probability that bidder i will accept the offer to buy the object, multiplying this by p we obtain the payoff of the winner from resale. The second term is her payoff, when $V_i < p$, and bidder i reject the offer.

Taking the first-order condition with respect to p we obtain

$$p - \frac{F_i\big(\phi_i(b)\big) - F_i(p)}{f_i(p)} = v_j. \tag{3}$$

It is assumed that F_i is regular, so the left-hand side is increasing, and consequently there exists a unique solution to (3), which is an optimal price $p_j(b, v_j)$ increasing both in b and v_j that maximizes bidder $j's$ payoff from resale.

The bidder $j's$ expected revenue from the resale (2), regardless it occurs or not, depends on b, v_j and $p_j(b, v_j)$, which in its turn is a function of the winning bid b and her own value v_j. Hence, the expected revenue of bidder j from resale is a composite function $R_j\big(p(b, v_j), b\big)$. In order to determine the change of the composite function $R_j(p(b, v_j), b)$ corresponding to the change in b Hafalir and Krishna implement the envelop theorem. Hence, the derivative of $R_j(p(b, v_j), b)$ with respect to b equals only to the direct effect of a change in b on the value of $R_j(p(b, v_j), b)$. Denoting the bidder $j's$ optimal expected revenue as $R_j(b, v_j)$ and taking the first-order condition with respect to b result in partial derivative

$$\frac{\partial}{\partial b} R_j(b, v_j) = f_i(\phi_i(b))\phi_i'(b)p_j(b, v_j) \qquad (4)$$

Bidding Stage

The bidding stage represents the actual auction, where both bidders are aware about the possibility of further resale, which means their bidding behavior is adjusted to the resale stage described above.

Suppose that, in equilibrium, both "weak" and "strong" bidder follows a continuous and strictly increasing bidding strategy $\beta_i: [0, a_i] \rightarrow R$, and each bidder choose such $\beta_i(v_i)$, when her value equals v_i. Similarly, as defined for the first-price sealed-bid auction, no bidder with value equaling 0 would submit positive bid, since it would result in loss, that is $\beta_s(0) = 0 = \beta_w(0)$. Moreover, there exists common highest submitted bid, such that $\beta_s(a_s) = \beta_w(a_w) \equiv \bar{b}$.

Let these equilibrium bidding strategies have inverses $\phi_i = \beta_i^{-1}$, such that $\phi_i: [0, \bar{b}] \rightarrow [0, a_i]$. Let b denote the winning bid of the auction, and suppose that $\phi_j(b) < \phi_i(b)$. Hence, should the bidder j win the auction, there would be potential gains from the resale and, consequently, the bidder j would offer to i to buy the object. However, should the bidder i win the auction, there would be no potential gains from resale, since the bidder i would value the object higher than j, and, consequently, no offer would be made.

The optimal strategy for bidder j is to bid b, since any deviation would decrease the expected payoff, so no other strategy could be optimal. The expected payoff of bidder j with value $v_j \equiv \phi_j(b)$, supposing that bidder i follows ϕ_i, from participating in the auction is

$$R_j(b, v_j) - F_i(\phi_i(b))b \tag{5}$$

The first term is the expected revenue from the resale, that is (2). The second term is the payment she has to make to the auctioneer when winning the auction. Taking the first-order condition to maximize (5), by using the result from the envelope theorem in (4), one obtains

$$0 = f_i(\phi_i(b))\phi_i'(b)p_j(b, v_j) - f_i(\phi_i(b))\phi_i'(b)b - F_i(\phi_i(b)),$$

$$(p_j(b, v_j) - b)f_i(\phi_i(b))\phi_i'(b) = F_i(\phi_i(b)),$$

$$\frac{f_i(\phi_i(b))\phi_i'(b)}{F_i(\phi_i(b))} = \frac{1}{p_j(b, v_j) - b},$$

where $p_j(b, v_j)$ is the optimal price of resale and is the solution to (2). Moreover, since $v_j \equiv \phi_j(b)$, p depends on b and $\phi_j(b)$ and can be defined as $p(b) \equiv p_j(b, \phi_j(b))$. So the first-order condition can be rearranged into the differential equation

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{p(b) - b}. \tag{6}$$

Similarly to the bidder j the optimal strategy for bidder i is to bid b, since any deviation would decrease the expected payoff, so no other strategy could be optimal. The expected payoff of bidder i with value $v_i \equiv \phi_i(b)$, supposing that bidder j follows ϕ_j, from participating in the auction is

$$(v_i - b)F_j(\phi_j(b)) + \int_{\phi_j(b)}^{a_j} [v_i - p_j(\beta_j(v_j), v_j)]_+ f_j(v_j)\, dv_j, \tag{7}$$

where $[x]_+ = \max\{x, 0\}$. The first term denotes the bidder $i's$ payoff when she wins the auction, since in this case she will not sell the object, it equals to payoff from ordinary first-price sealed-bid auction. The second term reflects the expected payoff of bidder i in the case of participating in resale and buying the object. Taking the first-order condition with respect to b resulting in

$$0 = [v_i - b]f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) -$$
$$- \left[v_i - p_j\left(b, \phi_j(b)\right)\right] f_j\left(\phi_j(b)\right) \phi_j'(b).$$

After simplifying

$$(p_j\left(b, \phi_j(b)\right) - b)f_j(\phi_j(b))\phi_j'(b) = F_j\left(\phi_j(b)\right),$$

$$\frac{f_j(\phi_j(b))\phi_j'(b)}{F_j\left(\phi_j(b)\right)} = \frac{1}{p_j\left(b, \phi_j(b)\right) - b}.$$

Expressing $p(b) \equiv p_j(b, \phi_j(b))$, the first-order condition can be rearranged into the differential equation

$$\frac{d}{db} \ln F_j\left(\phi_j(b)\right) = \frac{1}{p(b) - b}. \tag{8}$$

Hence, Hafalir and Krishna obtain the same differential equations, which determine optimal strategies for both “weak” and “strong” bidder. Since ϕ_s and ϕ_w are assumed to be inverse bidding strategies in equilibrium, they satisfy both equations (6) and (8).

Moreover, Hafalir and Krishna defined that despite the fact that initially bidders are asymmetric, in equilibrium the bid distributions of bidders are identical. This means that should ϕ_s and ϕ_w be increasing equilibrium inverse bidding strategies, then for any b

$$F_s\big(\phi_s(b)\big) = F_w\big(\phi_w(b)\big).$$

That is, the resale possibility symmetrizes the auction. Furthermore, “the distributions of equilibrium bids in the first-price auction with resale with asymmetric bidders are equivalent to the distribution of equilibrium bids in the first-price auction with symmetric bidders”.

The differential equations, which determine the equilibrium strategies for “strong” and “weak” bidder in the asymmetric first-price sealed-bid auction with resale, have the identical right-hand sides, which is contrasting to the result obtained in (1). The symmetrization occurs particularly due to the fact that both bidders have the same marginal gains from increasing their bids. This marginal gain equals to $p(b) - b$ and reflects the profit from resale for the “weak” bidder and the difference from the resale price in the case of losing the auction and the payment to the auctioneer in the case of winning for the “strong” bidder.

Conclusively, “the distributions of equilibrium bids in the first-price sealed-bid auction with resale under the assumption of asymmetry among bidders are equivalent to the common distribution of equilibrium bids in the first-price sealed-bid auction with symmetric bidders”. Hence, there exists such a distribution F, from which bidders in a symmetric first-price auction draw their values and which makes the asymmetric first-price sealed-bid auction with resale, where bidders draw their values from F_s and F_w, equivalent to the symmetric first-price sealed-bid auction. This means that

$$F_s\big(\phi_s(b)\big) = F\big(\phi(b)\big); \tag{9}$$

$$F_w\big(\phi_w(b)\big) = F\big(\phi(b)\big). \tag{10}$$

The equilibrium bidding strategy in the symmetric first-price sealed-bid auction is the same for both bidders, is similar to (1) and equals to

$$\frac{d}{db} lnF\big(\phi(b)\big) = \frac{1}{\phi(b) - b}. \tag{11}$$

Since ϕ_s and ϕ_w are supposed to be the inverse bidding strategies in equilibrium in the first-price sealed-bid auction with resale, using (9), (10) and (11) we could rearrange (6) and (8) to

$$\frac{d}{db} lnF_i\big(\phi_i(b)\big) = \frac{1}{\phi(b) - b}, \tag{12}$$

where $i = s, w$.

Hence, the right-hand sides of differential equations for equilibrium bidding strategies for "strong" and "weak" bidder are identical. Moreover, since we defined $p(b)$ to be the solution to (2), then for any b

$$p(b) = \phi(b).$$

Chapter 5

EQUILIBRIUM BIDDING STRATEGIES IN THE TWO-STAGE AUCTION WITH ASYMMETRIC BIDDERS

The model being under study is determined as follows and considers the two-stage auction of the type of the classical first-price sealed-bid auction in the first stage with the simultaneous bidding process and with sequential bidding in the second stage. The winning bid of the first stage, b, becomes publicly announced and is converted into the reserve price for the second stage. We assume the losing bid remains a private knowledge.

In the second stage, the winner of the previous round is assumed to move first. However, it is obvious that bidding just b is a dominant strategy for the winner of the first round to choose in the second stage. Hence, there is no incentive to deviate from the bid submitted in the first stage. In order to stimulate the winner of the first stage to increase her bid in the second stage, the auctioneer offers, in the case of losing the entire auction, to cover the half of the difference between her bids in both stages. After the first bidder submits her bid, the other bidder has two alternatives either to overbid the former or to reject competing for the object. The winner of the whole auction pays her own bid submitted in the second stage.

According to the auction theory, we are to solve the model described above by using the backward induction.

Suppose there are two risk-neutral bidders, whose values are independently, but asymmetrically distributed. Buyer $i's$ value of the object is distributed according to the distribution function F_i with support $[0, a_i]$ and with associated continuous density $f_i \equiv F_i'$. Buyer $j's$ value of the object is distributed according to the distribution function F_j with support $[0, a_i]$ and with associated continuous density $f_i \equiv F_j'$.

Suppose these two bidders follow bidding strategies β_i and β_j, which are assumed to be continuous and strictly increasing and have inverses ϕ_i and ϕ_j respectively, such that $\phi_i = \beta_i^{-1}$ and $\phi_j = \beta_j^{-1}$.

Suppose that bidder i with value v_i wins in the first stage with a bid b, which becomes the reserve price in the second stage. Then she has to move the first in the second stage and submit such $b_{i,2}$ that will maximize her expected payoff from the participation in the auction. Moreover, in the case when bidder i losses the entire auction, she receives the compensation from the auctioneer equaling to the half of the difference between her bid in the second stage, that is $b_{i,2}$, and the winning bid of the first stage, that is b.

The Second Stage

The bidder i has to choose such a $b_{i,2}$ that maximizes her expected payoff:

$$\max_{b_{i,2}} \left[(v_i - b_{i,2}) F_j(b_{i,2}) + \frac{b_{i,2} - b}{2} (F_j(\phi_j(b)) - F_j(b_{i,2})) \right] \qquad (13)$$

The term $F_j(b_{i,2})$ reflects the probability that the value of bidder j is less than $b_{i,2}$, which describes the situation when bidder i wins the auction and obtains as profit the difference between her own value and her bid made in the second stage. The second term describes the case, when the first stage winner losses the entire auction, but gets the payment from the auctioneer equaling the half of the difference between her bids.

Taking the first-order condition, we obtain:

$$0 = v_i f_j(b_{i,2}) - F_j(b_{i,2}) - b_{i,2} f_j(b_{i,2}) + \frac{F_j\left(\phi_j(b)\right)}{2} - \frac{F_j(b_{i,2})}{2} - \\ - \frac{b_{i,2} f_j(b_{i,2})}{2} + \frac{b f_j(b_{i,2})}{2},$$

$$0 = v_i f_j(b_{i,2}) - \frac{3}{2} F_j(b_{i,2}) - \frac{3}{2} b_{i,2} f_j(b_{i,2}) + \frac{F_j\left(\phi_j(b)\right)}{2} + \frac{b f_j(b_{i,2})}{2}$$

Rearranging the first-order condition for bidder $i's$ maximization problem, we obtain

$$b_{i,2} = \frac{2v_i + b}{3} + \frac{\frac{1}{3} F_j(\phi_j(b)) - F_j(\, b_{i,2})}{f_j(b_{i,2})}. \tag{14}$$

The optimal bid in the second stage $b_{i,2}(b, v_i)$ is a unique solution to the (14) and is an increasing function of b and v_i.

The First Stage

Suppose that every bidder i, in equilibrium, follows a continuous and strictly increasing strategy $\beta_i: [0, a_i] \rightarrow R$, which means that should the bidder $i's$ value be v_i, she submits bid equaling $\beta_i(v_i)$. Also suppose that there exists a common highest bid, such that $\beta_s(a_s) = \beta_w(a_w) \equiv \bar{b}$. Moreover, as defined above

$\beta_s(0) = 0 = \beta_w(0)$, since any bidder with value equaling 0 will incur losses, if submits any positive bid.

Let $\phi_i: [0, \bar{b}] \to [0, a_i]$ be $i's$ equilibrium inverse bidding strategy, which means that $\phi_i = \beta_i^{-1}$. Let b be the winning bid in the first stage. Should $\phi_i(b) < \phi_j(b)$ the bidder i would increase the bid in the second stage.

Suppose that bidder j follows her equilibrium bidding strategy ϕ_j. The bidder $i's$ expected payoff from participating in the auction, when her value is $v_i \equiv \phi_i(b)$, is just (13). When plugging in the solution obtained in (14) it results in

$$\max_b \left[v_i - \frac{2v_i + b}{3} - \frac{\frac{1}{3} F_j\left(\phi_j(b)\right) - F_j\left(b_{i,2}\right)}{f_j\left(b_{i,2}\right)} \right] \times$$
$$\times F_j\left(\frac{2v_i + b}{3} + \frac{\frac{1}{3} F_j\left(\phi_j(b)\right) - F_j\left(b_{i,2}\right)}{f_j\left(b_{i,2}\right)} \right) +$$
$$+ \frac{\frac{2v_i + b}{3} + \frac{\frac{1}{3} F_j\left(\phi_j(b)\right) - F_j\left(b_{i,2}\right)}{f_j\left(b_{i,2}\right)} - b}{2} \times$$
$$\times \left(F_j\left(\phi_j(b)\right) - F_j\left(\frac{2v_i + b}{3} + \frac{\frac{1}{3} F_j\left(\phi_j(b)\right) - F_j\left(b_{i,2}\right)}{f_j\left(b_{i,2}\right)} \right) \right) \quad (15)$$

The bidder $i's$ expected payoff from participating in the auction depends on b, v_i and $b_{i,2}(b, v_i)$, which in its turn is a function of the winning bid b and her own value v_i. Hence, the expected payoff of bidder i is a composite function, the change of which corresponding to the change in b could be determined by implementing the envelop theorem. Since the optimal bid in the second stage,

$b_{i,2}(b, v_i)$, and the function of bidder $i's$ expected payoff are differentiable, the derivative of (15) with respect to b equals only to the direct effect of a change in b on the value of (13), which results in

$$0 = \frac{b_{i,2}(b, v_i)}{2} f_j\left(\phi_j(b)\right)\phi_j'(b) -$$
$$-\frac{F_j\left(\phi_j(b)\right) + bf_j\left(\phi_j(b)\right)\phi_j'(b) - F_j\left(b_{i,2}(b, v_i)\right)}{2},$$

$$\left(b_{i,2}(b, v_i) - b\right)f_j\left(\phi_j(b)\right)\phi_j'(b) = F_j\left(\phi_j(b)\right) - F_j\left(b_{i,2}(b, v_i)\right).$$

Since $v_i = \phi_i(b)$, expressing $b_{i,2}(b, v_i) \equiv b_{i,2}(b)$, the first-order condition could be rearranged into the differential equation

$$\frac{d}{db}\ln F_j\left(\phi_j(b)\right) = \frac{F_j\left(\phi_j(b)\right) - F_j\left(b_{i,2}(b)\right)}{\left(b_{i,2}(b) - b\right)F_j\left(\phi_j(b)\right)}. \tag{16}$$

Suppose that bidder i follows ϕ_i. The optimal bidding strategy for bidder j in the first stage is to submit b, since any deviation would be unprofitable. Should the bidder $j's$ value be $v_j \equiv \phi_j(b)$ her expected payoff is

$$\left(v_j - b\right)F_i\left(\phi_i(b)\right) + \int_{\phi_i(b)}^{a_i} [v_j - b_{i,2}(\beta_i(v_i), v_i)]_+ f_i(v_i)\, dv_i, \tag{17}$$

where the first term reflects the case when bidder j wins in the first stage and, consequently the whole auction. When winning the first stage bidder j has no incentive to increase her bid in the second stage, since $\phi_i(b) < \phi_j(b)$, the bidder j wins the auction and pay just b, because bidder i rejects to overbid her in the second stage. The second term describes the case, when bidder j decides to participate in the second stage, and after submitting by bidder i her bid equaling

$b_{i,2}$, bidder j submits the same bid, wins the auction and gains the payoff of $\left(v_j - b_{i,2}(b, v_i)\right)$. Taking the first-order condition with respect to b we obtain

$$0 = v_j f_i\big(\phi_i(b)\big)\phi_i' - b f_i\big(\phi_i(b)\big)\phi_i' - F_i\big(\phi_i(b)\big) -$$
$$-\big[v_j - b_{i,2}\big(b, \phi_i(b)\big)\big] f_i\big(\phi_i(b)\big)\phi_i',$$
$$\big(b_{i,2}(b, \phi_i(b)) - b\big) f_i\big(\phi_i(b)\big)\phi_i'(b) = F_j\left(\phi_j(b)\right).$$

Expressing $b_{i,2}\big(b, \phi_i(b)\big) \equiv b_{i,2}(b)$, the first-order condition results in the differential equation

$$\frac{d}{db} \ln F_i\big(\phi_i(b)\big) = \frac{1}{\big(b_{i,2}(b) - b\big)}. \tag{18}$$

Since we claimed both ϕ_i and ϕ_j to be continuous and strictly increasing, solutions to (16) and (18) would reflect the equilibrium strategies of bidder j and bidder i. However, similarly to the first-price sealed-bid auction with asymmetric bidders, an explicit solution can be obtained only for some particular cases.

In contrast to the asymmetric first-price sealed-bid auction with resale, the differential equations for equilibrium strategies in the two-stage auction are not identical. Hence, the introduction of the second stage in the particular way described above does not symmetrize the first-price sealed-bid auction with asymmetric bidders.

Chapter 6

CONCLUSIONS

The classical auction theory operates under the number of assumptions, e.g. finite number of players, risk neutrality among bidders, independence of their private values, symmetry of bidders' values distributions etc. However, in reality, the majority of these assumptions do not hold at all. Hence, researchers in auction theory are aimed to improve existing models through adjusting them to the real world "imperfections".

In the thesis we release the assumption of symmetric bidders allowing for diversified players participating in the two-stage auction, which is similar to the model used in "ProZorro.Procurement", but simplified one with the ascending bidding process. The model investigated consists of the first stage with simultaneous bidding, which corresponds to the zero stage in "ProZorro", and of the second stage with sequential bidding, which in its turn corresponds to the three remaining stages in "ProZorro". In contrast to "ProZorro", the model described provides a reward for a bidder that wins the first stage but loses the entire auction. In order to put an incentive on the first stage winner to increase her bid in the second stage the auctioneer offers the half of the difference between two bids to remunerate.

We approach the determination of equilibrium strategies in the two-stage auction with asymmetric bidders through modifying the model of asymmetric first-price sealed-bid auction with resale proposed by Hafalir and Krishna (2008). Since the resale is similar to the sequential bidding at the second stage, we investigate the possibility of implementation the equilibrium obtained by Hafalir and Krishna (2008) into the model investigated in our research.

The main conclusion obtained by Hafalir and Krishna (2008) is that the possibility of resale incorporated into the asymmetric first-price sealed-bid auction symmetrizes bidders in terms of their equilibrium strategies. Moreover, despite the initial asymmetry, it is defined that bidders draw their bids from the same distribution.

In contrast to the equilibrium of the asymmetric first-price sealed-bid auction with resale, the equilibrium bidding strategies in the asymmetric two-stage auction are proved not to be identical. Despite the conceptual similarity of these two auctions, they differ in several aspects. In the model with resale, property rights on the object changes before the resale occurs, while in the two-stage auction the object remains a property of the auctioneer until the end of the second stage. So in the model described by Hafalir and Krishna (2008) the winner of the auction becomes the owner of the object and decides whether to sell this object to the loser. Hence, she bases her decision on comparison of the expected payoff from the resale and her own value of the object. However, in the two-stage auction, the winner of the first stage yet tries to get the object via participating in the second stage; and decides whether the reward from the auctioneer when losing the entire auction is higher than the expected payoff from winning the object. Hence, one of the main distinction of two models is change in the property rights on the object.

The equilibrium strategies of the two-stage auction are proved to exist, however, the explicit solution to the differential equations, which determines these strategies, can be found only in some cases. Hence, the comparison of optimal bids of the asymmetric first-price sealed-bid auction with resale with the asymmetric two-stage auction is possible only for the particular distributions, but not in the general case.

The equilibrium strategies obtained can be used for calculating the revenue of the auctioneer for some specific cases of distributions, however, calculations are supposed to be rather tedious. Once calculated the revenue, the further comparison of the asymmetric two-stage auction with the asymmetric first-price sealed-bid auction would become possible. Hence, the efficiency of more complicated model used in "ProZorro" in favor of plain first-price sealed-bid auction can be proved or rejected. Since the model described can be considered as a simplified proxy for the "ProZorro", the result obtained can be used in further theoretical researches aimed on theoretical proof of the efficiency of this auction system.

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