FIRST PRICE SEALED BID AUCTIONS WITH INTERDEPENDENT VALUATIONS

by

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A thesis submitted in partial fulfillment of
the requirements for the degree of

MA in Economic Analysis

Kyiv School of Economics

2014

Thesis Supervisor: ____________________ Professor Pavlo Prokopovych

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Abstract

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The focus of this research is on finding conditions for the existing of a pure and mixed strategy Nash equilibrium for the first price sealed bid auctions with interdependent valuations. First, we present a number of the sets of necessary and sufficient conditions under which a pure strategy Nash equilibrium exists in the first price sealed bid auction with complete information in the presence of externalities. Then, we study the conditions under which a mixed strategy Nash equilibrium exists in this type of auctions. In addition, we analyze the first price sealed bid auction with incomplete information and show the conditions in which symmetric mixed strategy Nash equilibrium exists.

To my parents

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ACKNOWLEDGMENTS

I wish to express his sincere gratitude to my thesis advisor, Professor Pavlo Prokopovych, for supervising of this research, overall guidance, valuable comments, interesting ideas, support and understanding.

I am grateful to Tom Coupe, Tetyana Dubovyk, Hanna Vakhitova and all the other professors who were participating in our Research Workshops for many helpful comments on my paper. In addition, I thank to all KSE professors, who are always open for questions and ready to help.

Special thanks goes to my classmate and friend Artur Grygorian for his support, teamwork and valuable advices.

In addition, I would like to thank all my classmates for their support and confidence in my ability to succeed in this theoretical research.

I thank my parents for their unconditional belief, encouragement and support.

Chapter 1

INTRODUCTION

The classical models of auctions analyze the situation when a bidder who loses does not care who wins the auction. However, many real life cases should be modeled as auctions with interdependent externalities. The presence of interdependent externalities means that the utility of a player who loses depends on that who wins the auction. For instance, if a museum wants to buy some famous painting in the auction and loses, its valuation may depend on that who wins: the other museum or a private collector. Another example is the competition between more than two oligopolists for a patent. In this case, it may be crucial for each of them who receive the patent, if they lose the competition (Funk, 1990, 1996). Baye, at al (1996) showed how the standard all pay auction models could be used to analyze various cases in different fields: political campaigns, job promotion, lobbying for rents, and tournaments (M. R. Baye et al, 1996). Thus, it is possible to notice that the theoretical analysis of such auctions can be successfully applied in practice for the analysis of different situations.

There are two main types of auctions, first price sealed bid auction and all pay sealed bid auction. First price sealed bid auction means that all players simultaneously submit their bids and after that, a player with the highest bid wins the auction paying her initial bid. All pay sealed bid auction is almost similar but at the end, all players pay their bids regardless of their winning.

A good example of application of the all pay auction model is elections. During campaigns, political powers spend millions of hryvnas to get as many voters as possible; however, there is only one winner. Increase the payments usually increases the chances to win. This situation can be modeled by using the standard all pay auction. However, in reality, the consequences for a big party, which lost the rayon elections, can be different. It depends on that who won: some local party, which is not even in the parliament, or their main opponent. Probably a major party prefers to lose to some local party, because this fact does not change the parity between the major parties in the country.

The results we intend to obtain in this theoretical research could be used for better understanding of the above situation, because they give more information about strategies and outcomes.

Thus, the main aim of the thesis is to study first price sealed bid auctions with interdependent externalities, and to find the conditions under which they possesses a Nash equilibrium, a stable point from no player wants to deviate given the other players' strategies.

The paper is organized as follows. Section 2 provides a short review of some related literature about the auctions without externalities and with externalities. Section 3 overviews the methodology, discusses some mathematical models of auctions. Section 4 presents a number of the sets of necessary and sufficient conditions under which a pure strategy Nash equilibrium exists in the first price sealed bid auction with complete information in the presence of externalities. Section 5 describes the conditions under which a mixed strategy Nash equilibrium exists in the first price sealed bid auction with complete information. Section 6 presents the conditions in which symmetric mixed strategy Nash equilibrium exists in the first price sealed bid auction with incomplete information. Section 7 concludes the results and suggests directions for future work.

Chapter 2

LITERATURE REVIEW

This section is organized as follows. The first part is concentrated on the methods and approaches used in the literature to model the auctions without externalities. The second part is focused on existing results about auctions with interdependent valuations.

2.1 Auctions without externalities

It should be noted that the question of equilibrium existence in the first price and all pay auctions without externalities has studied quite well:

Baye et al. (1996) proved that the mixed strategy equilibrium always exists in the model of all pay auctions without externalities. In this model, there is only one variable, which characterize every player. It is his or her valuation of the object. The authors considered all possible sets of combinations of valuations and find all sets of equilibriums in every case. The equilibriums described are as follows:

1. All valuations are equal. There is a unique symmetric equilibrium and a continuum of asymmetric equilibriums. In addition, it was founded that all these equilibriums are payoff equivalent meanings that the revenue for the seller is the same in every case.

2. The second and the third highest valuations are equal and strictly less than the first one. In this case there is a unique equilibrium in which all agents who have the same values use an identical strategy. In addition, there are a continuum of equilibriums without this condition. The interesting fact is that these equilibriums are not payoff equivalent. It

means that in different equilibriums the outcome for some player could be different.

3. At least the first three valuations are not equal to each other. This case was considered earlier by Hillman and Riley (1984). They found that there is a unique equilibrium in this situation.

Notice that Baye et al. (1996) fully characterized the set of equilibria in the all pay sealed bid auctions. In the model with externalities the number of possible cases is very large, because each player has not one characteristic (own valuation), but N (own valuation and $\mathrm{N}-1$ outcomes if some other player wins). However, the idea of partitioning into some classes can be successfully used in this case.

Albano and Matros (2004) considered a wider class of bidding games with complete information which include standard fist price sealed bid auctions and construct the continuum of pure and mixed strategy Nash equilibriums.
Thus, the issue about equilibria in the auction without externalities has been studied in much detail. Baye et al. (1996) described the set of equilibria of the all pay sealed bid auction, and Albano and Matros (2004) described the set for the first price sealed bid auction.

2.2 Auctions with externalities

Auctions with externalities have not been studied well enough. The general conditions in which the equilibrium exists still unknown. Only some special cases are considered. Let us consider the papers about auctions with externalities and examine special cases they analyzed.

Funk (1996) was the first to study the first price sealed bid auctions with externalities. The author considered the first price sealed bid auction two-steps model with auctioneer as a player and nonstandard tie-breaking rule, and found the equilibrium conditions. Tie-breaking rule determines the winner when several players ordered the highest bid. In this model, the auctioneer chooses the tie-breaking rule after all player submit his or her bids. Let us consider the model in detail:

An auctioneer would like to sell some object and there are some players who want to buy it. Every buyer submits her bid. The strategy for the seller is the function from the set of bids to the set of players which determine the winner. It is possible to assume that auctioneer bids zero because the auctioneer does not pay the price. The profit for the buyer is the difference between his valuation and bid if he wins, and some specific amount which depends of who wins if he loses. The profit for the auctioneer is also determined according to this concept. Let us introduce some additional notations: the maximum amount what player would like to pay in order to win rather than some other player wins. For the auctioneer it means the minimum price at which he or she would like sell the object. Funk proved that an equilibrium exists if there are two or more players who submitted the maximum amount and nobody wants to submit more in order to prevent the winning of one of these bidders.

However, in the conventional auctions, the auctioneer does not move after the start and the tie-breaking rule is also known for players before they bid. The setup of my model is different: the auctioneer is considered as a part of the game and the tie-breaking rule is common knowledge before the game started.

Another attempt to analyze first price sealed bid auctions with externalities was made Varma (2002). He considered the standard open and winner-pay sealed bid auctions with identity dependent externalities. However, Varma introduced only two types of externalities – zero for some players and some fixed negative amount for other. Thus, this model is considered as a special case of the model studied in this thesis.

Jehiel and Moldovanu (2006) showed that bidders might play significantly different equilibriums if externalities are present. However, they do not analyze the conditions in which these equilibria exist. They just consider some special case of auctions with and without externalities and show that in this case, equilibria exist and they are different. This underlines the importance of studying the model with externalities.

A number of attempts were made to analyze all pay sealed bid auctions with externalities. Konrad (2006) considered the model of all pay auction with only one bidder who has identity-dependent externalities. It means that only one player is not indifferent who wins the auction if he or she loses.

Sealed bid auctions with interdependent valuations were studied by Klose and Kovenock (2013), who proved that under very unrestrictive conditions in the all pay auction with externalities a mixed strategy Nash equilibrium always exists.

In this thesis, sufficient conditions for equilibrium existence in the first price sealed bid auction with interdependent valuations are investigated.

Chapter 3

METHODOLOGY

This section is organized as follows. The first part is concentrated on the methods and approaches used in the literature to model the auctions without externalities. The second part is focused on existing results about auctions with interdependent valuations.

P.Funk (1996) considered the first price sealed bid auction model with externalities. However, in this model the auctioneer determines the tie-breaking rule endogenously, in the course of the game. The model studied here uses the standard tie-breaking rule, which is more related to the reality, because in real life auctions auctioneer usually cannot make any decisions after the auction starts. This case has not been considered before.

To explain the model intuitively, let us consider the following situation. Suppose there are some political parties competing in an election. They can spend money to increase the loyalty of their electorate. Let us assume that the party which spends the biggest amount of money wins the elections. However, if a party loses the election, the spent money is not reimbursed. In addition, we assume that the payoff of the defeated party depends on who the winner is. This situation is described by the all pay sealed bid auction model with externalities.

Suppose there is a set of players $i \in I = \{1, \dots, n\}$. They compete for a single prize. Every player submits their bid $b_i \geq 0, i \in I$. The player with higher bid wins the auction. In the case of tie between k players with highest bids, everybody wins with probability $1/k$. This is the commonly used standard tie-breaking rule. Player i values the prize as $v_i, i \in I$, if she wins the auction. However, player's i valuation can depend on that who is the winner if she loses the auction. Let us denote these valuations as $h_i^j, j \neq i$, if player j wins the auction. Notice that h_i^j can be positive, negative or zero. All the information is common knowledge.

The player's utility functions in the model without externalities are:

$$U_i^{FP} = \begin{cases} v_i - b_i, & \text{if player } i \text{ wins the auction} \\ 0, & \text{if player j} \neq i \text{ wins the auction} \end{cases}$$

in the case of first price sealed bid auction.

$$U_i^{AP} = \begin{cases} v_i - b_i, & \text{if player } i \text{ wins the auction} \\ -b_i, & \text{if player j} \neq i \text{ wins the auction} \end{cases}$$

in the case of all pay sealed bid auction.

In order to realize the specification of my thesis research we are going to consider the following utility functions in the case with externalities:

$$U_i^{FPE} = \begin{cases} v_i - b_i, & \text{if player } i \text{ wins the auction} \\ h_i^j, & \text{if player j} \neq i \text{ wins the auction} \end{cases}$$

in the case of first price sealed bid auction.

$$U_i^{APE} = \begin{cases} v_i - b_i, & \text{if player } i \text{ wins the auction} \\ h_i^j - b_i, & \text{if player j} \neq i \text{ wins the auction} \end{cases}$$

in the case of all pay sealed bid auction.

The first model is adjusted to Funk's model with standard tie-breaking rule. The second model was proposed by B. Klose and D.Kovenock (2013).

In the thesis, the first price sealed bid auction with interdependent valuations is considered. First, a number of sets of sufficient conditions in which the pure strategy Nash equilibrium exists are found. Then conditions under which mixed strategy equilibria exist are studied. To facilitate studying the problem, a classification of different types of valuations of players is developed and then equilibrium existence conditions are given for each case.

In the thesis, we also consider the model where all players know their own valuation and externalities and have some beliefs about the other players' valuations and externalities. Thus, every player observes his or her own valuation and externalities before he/she bids only. The functional form of a symmetric mixed strategy Nash equilibrium is found.

Chapter 4

PURE STRATEGY NASH EQUILIBRIUM

Let us consider the first price sealed bid auction with identity-dependent externalities. To simplify mathematical derivations let us consider the case with three bidders. The method can be easily extended to the case of n bidders. Denote the player's preferences as

$$(v_1, h_1^2, h_1^3), (v_2, h_2^1, h_2^3), (v_3, h_3^1, h_3^2).$$

Assume that players bid b_1, b_2, b_3, respectively. Now let us consider all possible allocations of the bids and find the conditions for existing pure strategy Nash equilibrium in this model. Remember, that a situation is in equilibrium if nobody can deviate with profit.

I. $b_1 \neq b_2 \neq b_3$

There is no pure strategy equilibrium. Assume that $b_1 > b_2 > b_3$. The first player can always deviate to $b_1' \in (b_2, b_1)$. She still wins the game, but her utility will be higher. So, it is better to deviate.

II. $b_1 = b_2 \neq b_3$

If $b_3 > b_1 = b_2$, it is better for player 3 to deviate to some $b_3' \in (b_1, b_3)$. If $b_1 = b_2 > b_3$ the players' utilities according to the model are:

$$U_1 = \frac{1}{2}(v_1 - b_1) + \frac{1}{2}h_1^2$$

$$U_2 = \frac{1}{2}(v_2 - b_2) + \frac{1}{2}h_2^1$$

$$U_3 = \frac{1}{2}h_3^1 + \frac{1}{2}h_3^2$$

Now let us consider all possible deviations for every player and find the conditions in which these deviations are not profitable.

If player 3 deviates to $[0, b_1)$ nothing changes for him.

If player 3 deviates to b_1:

$$U_3' = \frac{1}{3}(v_3 - b_1) + \frac{1}{3}h_3^1 + \frac{1}{3}h_3^2$$

It should be that $U_3 \geq U_3'$.So,

$$\frac{1}{2}h_3^1 + \frac{1}{2}h_3^2 \geq \frac{1}{3}(v_3 - b_1) + \frac{1}{3}h_3^1 + \frac{1}{3}h_3^2 \tag{4.1}$$

If player 3 deviates to $(b_1, +\infty)$:

$$U_3'' = v_3 - (b_1 + \varepsilon)$$

It should be that $U_3 \geq U_3''$.So,

$$\frac{1}{2}h_3^1 + \frac{1}{2}h_3^2 \geq v_3 - b_1 \tag{4.2}$$

If player 2 deviates to $[0, b_2)$: $U_2' = h_2^1$.

It should be that $U_2 \geq U_2'$. So,

$$\frac{1}{2}(v_2 - b_2) + \frac{1}{2}h_2^1 \geq h_2^1 \tag{4.3}$$

If player 2 deviates to $(b_2, +\infty)$:

$$U_2'' = v_2 - (b_2 + \varepsilon).$$

It should be that $U_2 \geq U_2''$. So,

$$\frac{1}{2}(v_2 - b_2) + \frac{1}{2}h_2^1 \geq v_2 - b_2 \tag{4.4}$$

It is possible to write similar conditions for the first player:

$$\frac{1}{2}(v_1 - b_1) + \frac{1}{2}h_1^2 \geq h_1^2 \tag{4.5}$$

$$\frac{1}{2}(v_1 - b_1) + \frac{1}{2}h_1^2 \geq v_1 - b_1 \tag{4.6}$$

Notice that if inequalities (4.1) – (4.6) hold, nobody wants to deviate.

It is easy to find from (4.1) and (4.2) that

$$v_3 - \frac{1}{2}(h_3^1 + h_3^2) \leq v_1 - h_1^2 \tag{4.7}$$

from (4.3) and (4.4) that

$$b_2 = v_2 - h_2^1$$

and from (4.5) and (4.6) that

$$b_1 = v_1 - h_1^2$$

However, in the case first and second players should bid the same amount. So,

$$v_1 - h_1^2 = v_2 - h_2^1 \tag{4.8}$$

Thus, if conditions (4.7) and (4.8) hold, there are the pure strategy Nash equilibriums, such that player 1 bids $v_1 - h_1^2$, player 2 bids $v_2 - h_2^1$, and player 3 bids any number from $[0, v_1 - h_1^2)$.

Notice that conditions (4.7) and (4.8) are sufficient but not necessary. In order to find necessary conditions it is needed to consider one more case.

III. $b_1 = b_2 = b_3$

The utility for first player in this case is

$$U_1 = \frac{1}{3}(v_1 - b_1) + \frac{1}{3}h_1^2 + \frac{1}{3}h_1^3$$

If player 1 deviate to $[0, b_1)$:

$$U_1' = \frac{1}{2}h_1^2 + \frac{1}{2}h_1^3$$

It should be that $U_1 \geq U_1'$. So,

$$\frac{1}{3}(v_1 - b_1) + \frac{1}{3}h_1^2 + \frac{1}{3}h_1^3 \geq \frac{1}{2}h_1^2 + \frac{1}{2}h_1^3 \tag{4.9}$$

If player 1 deviate to $(b_1, +\infty)$:

$$U_1'' = v_1 - (b_1 + \varepsilon)$$

It should be that $U_1 \geq U_1''$. So,

$$\frac{1}{3}(v_1 - b_1) + \frac{1}{3}h_1^2 + \frac{1}{3}h_1^3 \geq v_1 - b_1 \tag{4.10}$$

It is easy to find from (4.9) and (4.10) that

$$b_1 = v_1 - \frac{1}{2}(h_1^2 + h_1^3) \tag{4.11}$$

Since the allocation is symmetric it is possible to write the similar conditions for player's 2 and player's 3 bids. So,

$$b_2 = v_2 - \frac{1}{2}(h_2^1 + h_2^3)$$

$$b_3 = v_3 - \frac{1}{2}(h_3^1 + h_3^2)$$

Remember, that $b_1 = b_2 = b_3$.

Thus, the condition in which the pure strategy Nash equilibrium exists in this case is:

$$v_1 - \frac{1}{2}(h_1^2 + h_1^3) = v_2 - \frac{1}{2}(h_2^1 + h_2^3) = v_3 - \frac{1}{2}(h_3^1 + h_3^2) \tag{4.12}$$

Therefore, it is possible to conclude that the pure strategy Nash equilibrium in the considered model exists if and only if conditions (4.7) and (4.8) hold or condition (4.12) holds. Notice that in pure strategy Nash equilibrium at least two players should bid the same amount.

Chapter 5

MIXED STRATEGY NASH EQUILIBRIUM

In this part let's consider the same model of the first price sealed bid auction with identity dependent externalities. We would try to analyze not only pure strategy equilibrium, but also mixed strategy Nash equilibrium in this model. Remember that $(v_1, h_1^2, h_1^3), (v_2, h_2^1, h_2^3), (v_3, h_3^1, h_3^2)$ is the set of valuations. Let us p_i^j is the maximum willingness to pay for player i in order to win rather than player j wins. It means that if player i bids p_i^j he is indifferent between winning and losing to player j. Thus, p_i^j should be equal to $(v_i - h_i^j)$. In order to better understand this important concept let us consider some numerical example. Let's$v_1 = 10, h_1^2 = -5, h_1^3 = -3$.

According to the notation, it means that the first player earns $10 - b_1$ if he bids b_1 and wins the auction, -5 if the second player wins, and -3 if player 3 wins. In this case $p_1^2 = 15, p_1^3 = 13$. Really, if player 1 bids 15 and wins, he earns $10 - 15 = -5$. If he lose for the second player he earns h_1^2, which is the same -5. Thus, if he bids 15 he is indifferent between losing for the second player and wining.

Now let's consider the set of the highest p_i^j,

$$S \coloneqq \left\{ p_i^j \middle| p_i^j = \max_{k,l} p_k^l \right\}$$

Notice that all elements in S are equal to the maximum willingness to pay among all participants.

Let us analyze all possible cases:

1. $\exists!\, i: p_i^j \in S$

It means that there is only one the highest p_i^j.

2. $\exists i, j: p_i^j \in S, p_j^i \in S$.

It means that there exist two players who dislike each other at the same highest level.

3. $\exists i, j:\ p_i^j \in S,\ p_j^k \notin S$ for all k.

It means that there is some player who are ready to pay the highest amount to prevent some other player to win but nobody dislike him at this level.

4. $\forall p_j^i \in S\ \exists k \neq i:\ p_j^k \in S$.

It means that there is some cycle with length more than 2. For example, the first player dislike the second player, the second player dislike the third player and the third player dislike the first one at the same level.
The main result is that in cases 1, 2, and 3 the mixed strategy Nash equilibrium always exists. The proof of this fact is quite complex and constructs an example of equilibrium in each case.

Case 1. $\exists!\, i: p_i^j \in S$.

In this case the following set of strategies will be the mixed strategy equilibrium: Player i bids $p_i^j - \varepsilon$, player j uniformly randomize between $p_i^j - \varepsilon - \delta$ and $p_i^j - \varepsilon$, and all other players bid zero.

Notice that in this case player i always wins the auction and her utility will be

$$U_i = v_i - \left(p_i^j - \varepsilon\right) = v_i - p_i^j + \varepsilon$$

For some little ε, $p_i^j - \varepsilon$ is higher than p_j^i, and thus there is no sense for player j to increase her bid. Now we need to demonstrate the appropriate δ, such that there is no sense for player i to decrease her bid. If player i decrease her bid to

$p_i^j - \varepsilon - \sigma, \sigma \geq \delta$, than player j wins for sure and the utility for player i in this case is h_i^j, which is less than $v_i - p_i^j + \varepsilon$.

If player i decrease her bid to $p_i^j - \varepsilon - \sigma, \sigma < \delta$ than her utility will be

$$U_i^* = \left(v_i - \left(p_i^j - \varepsilon - \sigma\right)\right)\frac{\delta - \sigma}{\delta} + h_i^j\frac{\sigma}{\delta}$$

The following condition should holds for every $\sigma > 0$: $U_i \geq U_i^*$.

In this case, there is no sense to deviate for any player and it means that the set of strategies is the mixed strategy Nash equilibrium.

$$v_i - p_i^j + \varepsilon \geq \left(v_i - \left(p_i^j - \varepsilon - \sigma\right)\right)\frac{\delta - \sigma}{\delta} + h_i^j\frac{\sigma}{\delta}$$

$$v_i - p_i^j + \varepsilon \geq \frac{v_i\delta - p_i^j\delta + \varepsilon\delta + \sigma\delta - v_i\sigma + p_i^j\delta - \varepsilon\sigma - \sigma^2 + v_i\sigma - p_i^j\delta}{\delta}$$

$$\sigma^2 - \sigma\delta + \sigma\varepsilon \geq 0 \;\Rightarrow \delta \leq \varepsilon$$

It means that for every $\varepsilon > 0$, such that $p_i^j - \varepsilon$ is higher than the highest element $p_k^n \notin S$, and for every $\delta > 0, \delta \leq \varepsilon$ the following set of strategies is a mixed strategy Nash equilibrium:

Player i bids $p_i^j - \varepsilon$, player j uniformly randomize her bid on $\left(p_i^j - \varepsilon - \delta, p_i^j - \varepsilon\right)$, and other players bid zero.

Case 2. $\exists i, j: p_i^j \in S, p_j^i \in S$.

In this case, the following set of strategies is the mixed strategy Nash equilibrium: player i bids p_i^j, player j bids p_j^i, other players bid zero. Notice that in this case the probability of winning for both players i and j is $\frac{1}{2}$ and for any other player is zero. The utility for player i in this case is

$$\mathrm{U_i} = \frac{1}{2}\left(\mathrm{v_i} - \mathrm{p_i^j}\right) + \frac{1}{2}\mathrm{h_i^j} = \mathrm{h_i^j}$$

If she decrease her bid, she lose for sure to player j and her utility is the same $\mathrm{h_i^j}$. If she increase her bid, she wins for sure and her utility is

$$\mathrm{v_i} - \left(\mathrm{p_i^j} + \varepsilon\right) = \mathrm{v_i} - \mathrm{p_i^j} - \varepsilon < \mathrm{h_i^j}$$

It means that there is no sense to deviate for player i. Similar arguments hold for player j. Notice that other players also cannot benefit from changing their bid, because their willingness to pay are not higher than $\mathrm{p_i^j}$. Thus, the situation is in equilibrium.

Case 3. $\exists \mathrm{i}, \mathrm{j}$: $\mathrm{p_i^j} \in \mathrm{S}$, $\mathrm{p_j^k} \notin \mathrm{S}$ for all k.

In this case the following set of strategies is the mixed strategy Nash equilibrium: player i bids $\mathrm{p_i^j} - \varepsilon$, player j uniformly randomize her bid on $\left(\mathrm{p_i^j} - \varepsilon - \delta, \mathrm{p_i^j} - \varepsilon\right)$, where $\varepsilon > 0$ is such that $\mathrm{p_i^j} - \varepsilon \geq \max\limits_{\mathrm{k}} \mathrm{p_k^i}$, and $\delta < \varepsilon$, and all other players bid zero.

In this case, player i wins for sure and enjoy utility

$$U_i = \mathrm{v_i} - \mathrm{p_i^j} + \varepsilon$$

Notice that for player i the situation is the same as in the case 1, and thus she cannot benefit from changing her bid. All other players also could not deviate with profit, because their willingness to pay in order to win rather than player i wins is lower than $\mathrm{p_i^j} - \varepsilon$. It means that the proposed set of strategies is the mixed strategy Nash equilibrium.

Case 4. $\forall p_j^i \in S\ \exists k \neq i\colon\ p_j^k \in S$.

The main result in this case is that there is no standard equilibrium. It means that there is no equilibrium than one players use pure strategy and other player randomize little bit less than this amount. Thus there is no equilibrium such that one player wins for sure. Let us prove this fact.

Let us assume that player i bids some amount b_i and wins for sure. There is 3 possible cases:

1. $b_i > v_i - p_j^i$
2. $b_i = v_i - p_j^i$
3. $b_i < v_i - p_j^i$

Let us consider each of them and prove that it is not an equilibrium.

In the first case, there is an incentive for player i to decrease her bid. Now she earn:

$$U_i = v_i - b_i < p_j^i$$

However, if she bids $v_i - p_j^i$ than her utility will be at least p_j^i. It means that player i could deviate with profit.

In the second case, the utility for player i is equal to

$$U_i = v_i - b_i = p_j^i$$

According to the assumption, the first player wins for sure, which means that no other players bid $v_i - p_j^i$. It means that the first player could decrease her bids and still has some chances to win. So, if she loses her utility will be at least p_j^i. However, if she wins she will pay less than $v_i - p_j^i$. We could conclude that the expected utility for player i will be higher in the case of deviation.

In the third case, let us consider utility for player z, who do not like player i. If player i wins for sure, than the utility for player z will be:

$$U_z = p_z^i = p_i^j$$

If player z bids some amount between b_i and $\mathrm{v_i - p_j^i}$ than the utility for this player will be:

$$U_z^* > \mathrm{v_i} - (\mathrm{v_i} - \mathrm{p_j^i}) = p_i^j$$

We could see that in this case, there is a player who can profitable deviate. Thus, this situation is not in equilibrium.

As we show above, there is no such equilibrium where some player wins for sure. The existence of some other non-standard types of equilibria could be studied in future research.

Chapter 6

AUCTIONS WITH INCOMPLETE INFORMATION

In this part, let us consider the same model of the first price sealed bid auction with identity dependent externalities. Remember, that in this model, every player knows all valuations and externalities. Now we are going to consider slightly different model. In contrast to the previous part, every player knows only the distribution functions of the opponents' valuations and externalities. Let us assume that for every player the own valuation has uniform distribution from one to two, and externalities have uniform distribution from zero to one. Thus, we assume that in the case of winning player always earns more compared to the case of losing, which is logical. Strategy in this model is some function from player's valuation and externalities to player's bid. Let us find a symmetric equilibrium in this model, because it is a good point to start. It means that every player should use the same strategy to identify their bid. For simplicity, let us consider the case with three players.

Let us denote the player's preferences as

$$(v_1, h_1^2, h_1^3), (v_2, h_2^1, h_2^3), (v_3, h_3^1, h_3^2)$$

According to the assumptions:

$$v_i \in U[1,2], \qquad h_j^k \in U[0,1], \qquad i, j, k \in \{1,2,3\}, j \neq k$$

Let us assume that $\beta(\cdot,\cdot,\cdot)$ is a symmetric equilibrium. Thus,

$$\beta: (v_1, h_1^2, h_1^3) \to b_1$$

is a first player's bid.

Notice that the first player wins only if he bids the highest amount. In this case he earns $(v_1 - b_1)$. If he loses the auction, his expected payoff is

$\frac{1}{2}(h_1^2 + h_1^3)$ by symmetry, because the probability for player 1 to lose to player 2 is the same as to lose to player 3. The probability of losing is

$$P(b_1 < \max(b_2, b_3))$$

Thus, the expected payoff for player 1 is

$$E_1(b_1) = (v_1 - b_1)P(b_1 > \max(b_2, b_3)) \\ + \frac{1}{2}(h_1^2 + h_1^3)P(b_1 < \max(b_2, b_3))$$

Notice that

$$P(b_1 = \max(b_2, b_3)) = 0,$$

because the player use mixed strategy and the probability that the bid will be equal to some fixed amount is equal to zero.

Let us rewrite:

$$E_1(b_1) = (v_1 - b_1)P(b_1 > \max(b_2, b_3)) + \\ + \frac{1}{2}(h_1^2 + h_1^3)(1 - P(b_1 > \max(b_2, b_3))) \tag{6.1}$$

$$E_1(b_1) = \left(v_1 - \frac{1}{2}(h_1^2 + h_1^3) - b_1\right)P(b_1 > \max(b_2, b_3)) \\ + \frac{1}{2}(h_1^2 + h_1^3) \tag{6.2}$$

Remember that player 1 chooses b_1 to maximize $E_1(b_1)$, which is his expected payoff. Notice that for any (v_1, h_1^2, h_1^3) and $(v_1^*, h_1^{2^*}, h_1^{3^*})$ such that

$$\left(v_1 - \frac{1}{2}(h_1^2 + h_1^3)\right) = \left(v_1^* - \frac{1}{2}\left(h_1^{2^*} + h_1^{3^*}\right)\right)$$

the optimal b_1 should be the same. Thus, the best bid for player 1 depends only on the

$$\left(v_1 - \frac{1}{2}(h_1^2 + h_1^3)\right)$$

In this case it is needed to find the symmetric equilibrium in the following form:

$$\beta: (v_1 - \frac{1}{2}(h_1^2 + h_1^3)) \to b_1$$

which significantly simplifies the problem, because now the domain of the function is the set in R, not in R^3.

Let us denote

$$u_i := v_i - \frac{1}{2}\left(h_i^j + h_i^k\right), \qquad i, j, k \in \{1,2,3\}$$

Every player knows only his or her own u. This variable has some distributed function F, which is possible to find, because the distribution functions for v_i, h_i^j, and h_i^k , $i, j, k \in \{1,2,3\}$ are known. Now it is possible to implement the method described by P. Monteiro (2006).

Let us consider the general case and then find the equilibrium in the specific problem. Let's $Y^{n-1}$is a maximum of $(n-1)$ independently distributed variables $u_2, \dots, u_n$. Thus, the distribution function of Y^{n-1} is

$$G(u) = F^{n-1}(u)$$

and the density of $G(u)$ is

$$g(u) = (n-1)F^{n-1}(u)f(u)$$

where $f(u)$ is a density of $F(u)$.

1. Let us prove that $\beta(0) = 0$.

Notice that $\beta(0)$could not be negative, because bids should be also positive or zero. If $\beta(0) > 0$ than

$\beta(\varepsilon) > \beta(0) > 0$, for any $\varepsilon > 0$.

Thus, for some ε:

$$\left(\varepsilon - \beta(\varepsilon)\right) < 0$$

$$P\big(\beta(\varepsilon) > max(b_2, \dots, b_n)\big) > 0$$

So,

$$E_1\big(\beta(\varepsilon)\big) < E_1(0)$$

It means that if player bids zero, the expected payoff is higher compare to the case if he or she bids $\beta(\varepsilon)$. Contradiction, because $\beta(\cdot)$ should be an equilibrium strategy and maximize the expected payoff.

2. Remember that

$$E_1(b_1) = (u_1 - b_1)P(b_1 > \max(b_2, b_3)) + C$$

where C is a constant. Let us rewrite it with new variables:

$$E_1(b_1) = (u_1 - b_1)P\big(Y^{n-1} \le \beta^{-1}(b_1)\big) + C = (u_1 - b_1)G\big(\beta^{-1}(b_1)\big) + C$$

The necessary conditions for the equilibrium strategy is the following:

$$\frac{d\big(E_1(b_1)\big)}{db_1} = 0.$$

Let us take a derivative and solve the following differential equation:

$$-G\big(\beta^{-1}(b_1)\big) + \frac{(u - b_1)g\big(\beta^{-1}(b_1)\big)}{\acute{\beta}\big(\beta^{-1}(b_1)\big)} = 0,$$

$$G\big(\beta^{-1}(b_1)\big)\acute{\beta}\big(\beta^{-1}(b_1)\big) = (u - b_1)g\big(\beta^{-1}(b_1)\big), \;\; b_1 = \beta(u)$$

$$G(u)\acute{\beta}(u) = \big(u - \beta(u)\big)g(u) \Leftrightarrow G(u)\acute{\beta}(u) + \beta(u)g(u) = ug(u)$$

Thus,

$$\big(G(u)\beta(u)\big)' = ug(u).$$

In order to solve this equation, let us integrate it:

$$G(u)\beta(u) = \int_0^u yg(y)dy \;\Rightarrow$$

$$\beta(u) = \frac{1}{G(u)}\int_0^u yg(y)dy \tag{6.3}$$

Remember, that $\beta(\cdot)$ is a symmetric equilibrium strategy. Condition (6.3) is a necessary condition for that. It is easy to show that every function which satisfies (6.3) is a symmetric equilibrium in the model.

Chapter 7

CONCLUSION

This theoretical research has been made to expand knowledge about auctions with interdependent valuations. Studying of this type of auctions could help better modeling a lot of real life situation, because it is very common that for someone who loses the auction it is important who the winner is. All pay sealed bid auction was considered by Klose and Kovenok (2013), that why we decided to focus on the first price sealed bid auctions. There are three main results obtained in this research.

A number of the sets of necessary and sufficient conditions under which a pure strategy Nash equilibrium exists in the first price sealed bid auction with complete information in the presence of externalities were found. This result is presented in section 4.

Also, the conditions under which a mixed strategy Nash equilibrium exists in the first price sealed bid auction with complete information were obtained. To fully examine this type of auction it is needed to consider the case 4 from section 5. It could be an opportunity for future research.

In addition, we studied the case of incomplete information and found the conditions in which symmetric mixed strategy Nash equilibrium exists. The next step in this direction could be a description of all asymmetric equilibria in the model.

WORKS CITED

Albano, Guan Luidgi, and Alexander Matros. 2004. "(All) Equilibria in a class of bidding games", *Economics Letters*, Volume 87, Issue 1, April 2005, Pages 61–66

Baye, Michael, Dan Kovenock, and Casper G. de Vries. 1996. "The all-pay auction with complete information", *Economic Theory* 8: 291 – 305

Funk, Peter. 1990. "The persistence of monopoly and the direction of technological change", Mimeo University of Bonn

Funk, Peter. 1996. "Auctions with Interdependent Valuations", International *Journal of Game Theory* 25: 51 – 64

Jehiel, Philippe, and Benny Moldovanu. 2006. "Allocative and informational externalities in auctions and related mechanisms", In: Blundell R, Newey WK, Persson T (eds) Advances in Economics and Econometrics: Volume 1: Theory and Applications, Ninth World Congress, vol. 1, Cambridge University Press, chap 3

Klose, Bettina, and Dan Kovenock. 2013. "The all-pay auctions with complete information and identity-dependent externalities", Working paper

Konrad, Kai A. 2006. "Silent interests and all-pay auctions", *International Journal of Industrial Organization* 24(4):701-713

Monteiro, Paulo Klinger. 2006. "First-price auction symmetric equilibria with a general distribution", *Games and Economic Behavior* 65 (2009) 256–269

Prokopovych, Pavlo. 2011. "On equilibrium existence in payoff secure games", *Economic Theory* volume 48, issue 1, pp 5-16

Prokopovych, Pavlo. 2013. "The single deviation property in games with discontinuous payoffs", *Economic Theory* 53, 383–402.

Prokopovych, Pavlo, and Nicolas C. Yannelis. 2014. "On the existence of mixed strategy Nash equilibria", *Journal of Mathematical Economics* 52 (2014) 87–97

Varma, Gopal das. 2002. "Standard auctions with identity-dependent externalities", *The RAND Journal of Economics* 33(4):689–708