STATIONARY EQUILIBRIA IN A BARGAINING MODEL

by

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A thesis submitted in partial fulfillment of
the requirements for the degree of

Master of Arts in Economics

National University "Kiev-Mohyla
Academy"

2000

Approved by ____
Chairperson of Supervisory Committee

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Program Authorized
to Offer Degree ____

Date ____

National University "Kiev-Mohyla Academy"

Abstract

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We apply a cost-saving approach to studying a number of mutual defense games. This approach is founded on computing cost savings from interior borders of the members of an alliance. We show that location vis-à-vis potential allies and border attributes matters for alliance formation and burden sharing if an alliance were to form. A "NATO-Ukraine-Slovakia" mutual defense game is studied both in the short- and in the long run.

The course of action countries should follow to achieve a mutually acceptable utility distribution is under study. We prove the existence of a stationary strategy equilibrium in an n-person bargaining game when players'

utility functions are concave.

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ACKNOWLEDGMENTS

I wish to express my deep gratitude to my supervisors Prof. Roy Gardner of Indiana University and Prof. Larysa Krasnikova of the University of Kiev-Mohyla Academy for their help and advice. I am also grateful to Prof. Marco Mariotti of the University of Exeter and Prof. Eyal Winter of Hebrew University of Jerusalem who greatly contributed to developing my interest in bargaining theory problems and advised me on numerous occasions. I owe much gratitude to EERC MA Program Director Michael Blackman and Prof. Charles Steele who provided me with much encouragement and support.

GLOSSARY

Discount factor: a factor used to compare the value of a dollar received in the future to a dollar received today.

Extensive-form representation of a game: one that specifies: (1) the players in the game, (2a) when each player has the move, (2b) what each player can do at each of his or her opportunities to move, (2c) what each player knows at each of his or her opportunities to move, and (3) the payoff received by each player for each combination of moves that could be chosen by the players.

Games with complete information: games in which the players' payoff functions are common knowledge

Games with perfect information: at each move in the game the player with the move knows the full history of the play of the game thus far.

Pure strategy Nash equilibrium: a profile of pure strategies from which no player can obtain a higher expected payoff by deviating.

Nash equilibrium: in a Nash equilibrium, each player's equilibrium action is at least as good as very other his action, given the other players' actions.

Noncooperative game theory: it focuses on how cooperation may emerge as rational behavior in the absence of an ability to make binding agreements.

Stationary equilibria: equilibria in which the players' moves are independent of the time period and history of the game.

Strategic game (normal-form): a game that consists of a set of players; for each player, a set of actions; for each player, a preference relation over the set of action profiles.

Strategy: is a complete contingent plan, or decision rule, that specifies how the player will act in every possible distinguishable circumstance in which he might be called upon to move.

Subgame perfect equilibrium: a game equilibrium in which every player plays an equilibrium on every subgame.

Subgame: a subgame in an extensive-form game (a) begins at a decision node n that is a singleton information set (but is not the game's first decision node), (b) includes all the decision and terminal nodes following n in the game tree (but no nodes that do not follow n) and (c) does not cut any information sets

Symmetric game: each player has the same set of strategies and every pair of players have the same utility function in the sense that, given the strategies of all the other players, interchanging the strategies of players interchanges their payoffs.

n **-person coalitional game with transferable utility:** (N, v), $N = \{1, \ldots, n\}$ is the set of players; $v : \Sigma \to R$ is the characteristic function acting from the set of all coalitions Σ.

Strictly convex bargaining game in characteristic form (N, v) **:** if for all $S, T \subset N$, with $S \setminus T$ and $T \setminus S$ nonempty, $v(S \cup T) > v(S) + v(T) - v(S \cap T)$.

Chapter 1

INTRODUCTION

Bargaining is of interest to economists not merely because many transactions are negotiated but also because conceptually two-person bargaining is somewhat the opposite of perfect competition among infinitely many traders. Although it is not difficult to understand that the outcomes of successful bargaining should lie in a contract curve, the main problem is to predict to which particular outcomes bargaining will lead.

Background

At present, there are two major approaches to studying bargaining problems: Nash's axiomatic approach (1953) and Rubinstein's strategic approach (1982). Nash's axiomatic solution to a bargaining game has advantages that are hard to exaggerate: the solution defined by the axioms is unique and its simple form is easily tractable.

In the strategic approach, the outcome is an equilibrium of an explicit model of bargaining process. The Rubinstein infinite horizon, alternating offers bargaining model has been considered as the fundamental extensive form for non-cooperative bargaining games. The most prominent feature of the model is that it provides a unique subgame-perfect equilibrium solution, which moreover is efficient.

The model has been extended to accommodate three or more players. The case when each player in an n-person bargaining game had a veto (the most direct generalization of Rubinstein's two-person bargaining model) was studied by Herrero (1985). If some of the players lack a veto then coalitions with fewer than all players may reach an agreement, which complicates the game a lot. The notion of the core comes to the rescue here. In the core, no coalition can create a better deal for itself than that resulting from overall cooperation. The core turned out to be indispensable in studying defense economics, especially the formation and burden sharing (Gardner, 1995). Gardner's ideas were applied to studying the current NATO expansion processes by Sandler (1999).

Having determined the core of a cooperative mutual defense game, we face another problem: that of describing the process of bargaining that results in an outcome belonging to the core. It can be showed that for a number of coalitional bargaining games any payoff vector in the core of the underlying game is the outcome of some stationary subgame-perfect strategy equilibrium of the bargaining game without discounting (Chatterjee *et al.*, 1993; Evans, 1997). On the other hand, Chatterjee *et al.* (1993) showed that for some strictly convex bargaining games as the discount factor tends to 1, the corresponding sequence of efficient stationary equilibria converges to a point in the core. So the problem of the existence of a stationary strategy equilibrium is akin to the problem of the nonemptiness of the core in this game. For a broad class of n-person infinite horizon bargaining games. Herrero (1985) and Chatterjee *et al* (1993) showed that there is no hope of obtaining the core without an assumption of stationarity. Evans (1997) gave an example of a natural noncooperative discrete-time noncooperative coalitional bargaining game that yields the core as its pure stationary subgame perfect equilibrium payoff set. In the case of continuous time, a model possessing this property was proposed by Perry and Reny (1994). As a result of these studies, interest in the problem of existence of a stationary strategy equilibrium has increased recently (Banks and Duggan, 1998; Ray and Vohra, 1999). Note that among the first papers devoted to the existence problem in economics were those by Debreu (1952) and Glicksberg (1952). These papers' results have been used by a large number of economists until now. Other considerable contributions to studying the problem of existence of an equilibrium in economic problems were the papers by Ichiishi (1981) and Dasgupta-Maskin (1986).

Ray and Vohra (1999) proved that a stationary subgame perfect equilibrium exists in a coalitional bargaining game in characteristic form if the only source of mixing is in the probabilistic choice of a coalition by each proposer. Their reasoning was based on the fact that stationary equilibria turned out to be analytically tractable in characteristic form bargaining games (they are solutions to a system of equations).

On the other hand, the very problem was also studied for a bargaining model of social choice by Banks and Duggan (1998). Having assumed that players' utility functions are strictly concave, these authors showed that in a majority bargaining game with a random recognition rule there exists a mixed strategy stationary equilibrium. Another important contribution to studying coalitional bargaining with random proposers was Okada's 1996 paper, in which he studied the properties of the set of stationary subgame perfect pure strategy equilibria in a bargaining game in characteristic form. A bargaining model congenial to those studied in the two latter papers is under investigation in Chapter 3 of this thesis.

Outline of the Thesis.

Chapter 2 contains a discussion of a number of mutual defense models that have arisen in the course of analyzing a number of possible trends of NATO expansion from Ukraine's point of view. To facilitate the reader's understanding, Gardner's benchmark mutual defense game is described. After that, mutual defense games are studied for different mutual locations of would-be members of an alliance. The proposed models allowed us to study a "NATO-Slovakia-Ukraine" game. Both in the short run and in the long run the center of gravity of the core is explored, that is the most appealing solution to the mutual defense game. It is shown that cost savings resulting from NATO expansion are more equally distributed in the long run than in the short run. In other words, the objections to NATO expansion regarding unequal burden sharing that can arise in the short run, will fade away in the long run.

In Chapter 3, the problem of existence of a stationary equilibrium is studied in an n-person bargaining game where players' utility function are concave (a player's attitude to risk is neutral or averse). An original way to determine a stationary strategy equilibrium is developed. In order to make invoking Glicksberg's results possible, the upper and lower semicontinuities of a number of set-valued maps are investigated.

Chapter 4 contains the main conclusions of this work.

Chapter 2

MUTUAL DEFENSE GAMES

For more than 40 years Europe was divided in two adversary groups of countries. Two military blocks were set up: NATO in 1949 and WTO (the Warsaw Treaty Organization) in 1955. At that time, countries making a decision whether to enter an alliance took into account mainly political not cost-sharing considerations. So, the forces that the communist countries allied with the Soviet Union contributed to the WTO were military weak and possibly unreliable in combat. In the case of a conventional European interblock war, the Soviet military would have scored as well without its allies. As time showed, among the major functions of the WTO was that of creating a deterrent against East European nationalist thinking and the institutional frameworks for Soviet military intervention in East European countries when deemed necessary. On the other hand, NATO was formed in 1949 as a counter to Soviet aggression in Eastern Europe and its takeover of satellite states.

With the end of the cold war and the collapse of the Soviet Union, new rationales for NATO's exis

(the benchmark case, Sandler (1999)) was proposed by Gardner (1995). In this model, spatial and locational characteristics of the countries are regarded as decisive in identifying the gains from mutual defense. A few preliminary definitions and concepts of cooperative game theory are required to formalize the game of mutual defense.

Preliminaries: the Benchmark Game

Here is a numerical example with three countries.

Example 1 (Gardner, 1995, p. 401) . Three contiguous countries are contemplating a mutual defer

the area being defended. The assumption is appropriate since a country of the size of Ukraine needs more forces for security than, for example, Moldova, no matter where the forces are stationed.

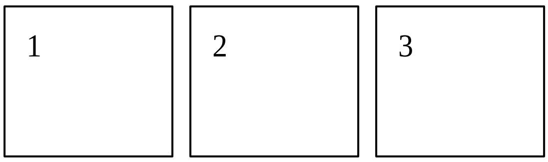

Figure 1. The Benchmark Mutual Defense Game.

Each side of the countries depicted is assumed to cost 1 to defend, so that $c(\{i\})$, the cost of defending nation i is equal to 4. Similarly, $c(T)$ represents the cost of protecting an alliance of countries T. We assume that the motivation for forming an alliance is cost savings as borders become interior and no longer need protecting. As a result, $c(\{1,2\}) = 6$, $c(\{1,3\}) = 8$, $c(\{2,3\}) = 6$, $c(\{1,2,3\}) = 8$.

If the three countries form a coalition than the global cost saving is as follows:

$c(\{1\}) + c(\{2\}) + c(\{3\}) - c(\{1,2,3\}) = 4$.

Let the countries share it equally among themselves, than their costs are the following ones:

$x_1 = 4 - 1.33 = 2.77$, $x_2 = 2.77$, $x_3 = 2.77$. Since $x_1 + x_2 < c(\{1,2\})$, $x_2 + x_3 < c(\{2,3\})$, $x_1 + x_3 < c(\{1,3\})$, there is no incentive for any pair of the countries to form a coalition not including the third country (in this case, their costs are higher).

The above reasoning can be readily formalized with the help of the core idea, which has sometim total cost met by a coalition should never exceed the cost incurred by this coalition if it were to defend itself:

$$\text{for all } T \subset N: \sum_{i \in T} x_i \leq c(T). \tag{2.1}$$

Definition 1. (Moulin, 1991, p. 91) . Given is a cost-sharing game (N, c), where

$N = \{1, 2, \ldots, n\}$ is the set of players, and c associates to each coalition T of N its cost $c(S) \geq 0$. The *core* of (N, c) is the set of cost allocations x satisfying $\sum_{i=1}^{n} x_i = c(N)$ and property (2.1).

Let us compute the core cost allocations in the example. Property (2.1) is the following system of i

$$x_1 + x_2 + x_3 = 8, \ \ x_1 \leq 4, \ x_2 \leq 4, \ x_3 \leq 4,$$

$$x_1 + x_2 \leq 6, \ x_1 + x_3 \leq 8, \ x_2 + x_3 \leq 6.$$

In order to facilitate visualizing the solution to the system, let us define player i 's cost savings as $y_i = c(i) - x_i$. Then

$$y_1 + y_2 + y_3 = 4, \ y_i \geq 0, \ i = 1,2,3,$$

$$y_1 + y_2 \geq 2, \ y_2 + y_3 \geq 2, \ y_1 + y_3 \geq 0.$$

The simplex $\{y_1 + y_2 + y_3 = 4, \ y_i \geq 0, \ i = 1,2,3\}$ is drawn in Figure 2.

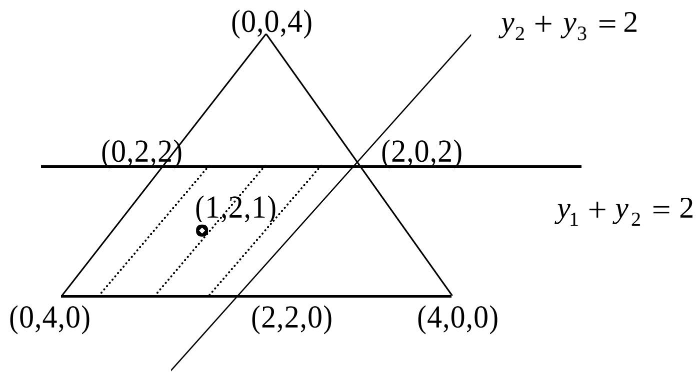

Figure 2. The Core: the Benchmark Mutual Defense Game.

The shaded rhombus with vertices $(0,2,2)$, $(2,0,2)$, $(0,4,0)$, $(2,2,0)$ represents the core of the game. The most appealing solution to the cooperative game theoretic problem is the center of gravity of the rhombus, which gives country 2 a bigger share of cost savings. The game representation is a simplified one, but we further add realism to it.

Several Generalizations of the Benchmark Mutual Defense Game

A number of generalizations of the model were proposed by Sandler (1999). He showed that withi effort on studying the opportunity of allying Ukraine with some or other of her neighbors.

All the following cases are caused by real life considerations. For example, studying a would-be coa

Example 2. The Case of Internal Boundaries of Different Length.

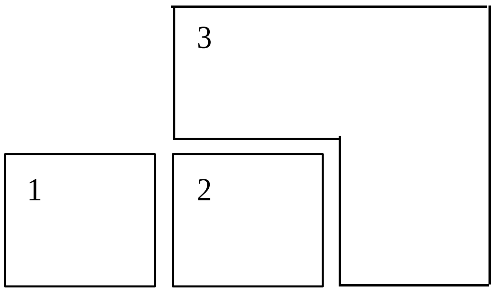

Figure 3. The Case of Internal Boundaries of Different Length.

The cost of protecting both country 1 and country 2 is equal to 4, that of protecting country 3, $c($·

$$x_1 + x_2 + x_3 = 10 \quad x_1 \le 4, \; x_2 \le 4, \; x_3 \le 8,$$

$$x_1 + x_2 \le 6, \; x_1 + x_3 \le 12, \; x_2 + x_3 \le 8.$$

Changing variables ($y_i = c(i) - x_i$),

$$y_1 + y_2 + y_3 = 6, \; y_i \ge 0, \; i = 1,2,3,$$

$$y_1 + y_2 \ge 2, \quad y_2 + y_3 \ge 4, \; y_1 + y_3 \ge 0.$$

The solution to this system is given in Figure 4.

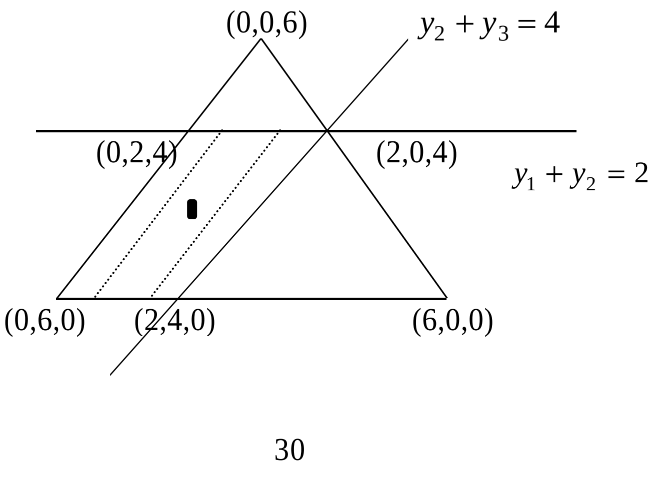

Figure 4. The Core: the Case of Internal Boundaries of Different Length.

As in the benchmark case, the most appealing solution to the cooperative game is the center of gravity $(1,3,2)$ of the rhombus with vertices (0,5,0), (1,4,0), (1,1,2), (0,2,2). That is, the interior country receives the biggest share of cost savings in the case as well.

Example 3. An essential difference of this case from the previous one is that country 3 has also a co

is equal to 443 km. Another example of this type is a coalition including Ukraine, Romania and Moldova.

Here we study the two following possible country locations:

(A)

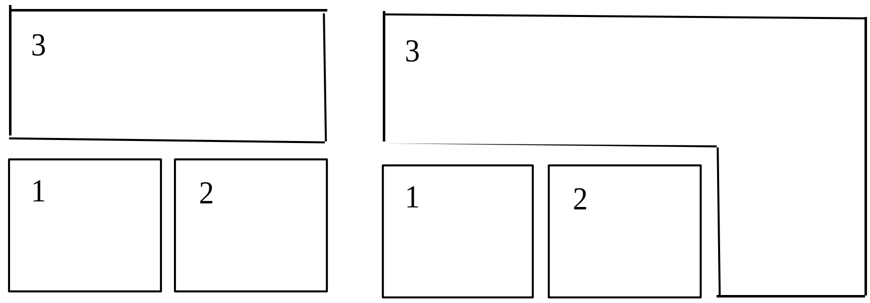

Figure 5. Two Cases of Pairwise Common Boundaries

(A) $c(\{1\}) = c(\{2\}) = 4$, $c(\{3\})$ =6. In this case, $c(\{1,2\}) = 6$, $c(\{1,3\}) = 8$, $c(\{2,3\}) = 8$, $c(\{1,2,3\}) = 8$. Property (2.1) gives us:

$$x_1 + x_2 + x_3 = 8 \quad x_1 \le 4,\ x_2 \le 4,\ x_3 \le 6,$$

$$x_1 + x_2 \le 6,\ x_1 + x_3 \le 8,\ x_2 + x_3 \le 8.$$

Changing variables ($y_i = c(i) - x_i$),

$$y_1 + y_2 + y_3 = 6,\ y_i \ge 0,\ i = 1,2,3,$$

$$y_1 + y_2 \ge 2,$$

$y_2 + y_3 \ge 2$, $y_1 + y_3 \ge 2$. The solution to this system is given in Figure 6.

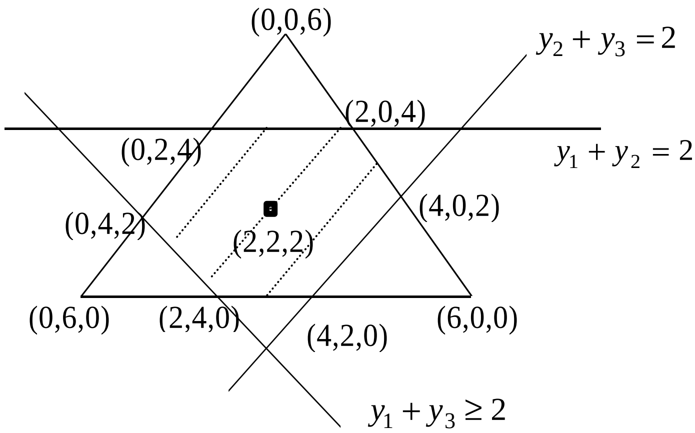

Figure 6. The Core: Case (A)

The center of gravity of the core (2,2,2) is the most appealing solution to the cooperative game, which corresponds to the fact that each of the countries has common boundaries of a length of 2 with the two other countries.

(B) $c(\{1\}) = c(\{2\}) = 4$, $c(\{3\}) = 10$. In this case, $c(\{1,2\}) = 6$, $c(\{1,3\}) = 12$, $c(\{2,3\}) = 10$, $c(\{1,2,3\}) = 10$. Property (2.1) gives us:

$$x_1 + x_2 + x_3 = 10 \quad x_1 \leq 4,\ x_2 \leq 4,\ x_3 \leq 10,$$

$$x_1 + x_2 \leq 6,\ x_1 + x_3 \leq 12,\ x_2 + x_3 \leq 10.$$

Changing variables ($y_i = c(i) - x_i$),

$$y_1 + y_2 + y_3 = 8,\ y_i \geq 0,\ i = 1,2,3,$$

$$y_1 + y_2 \geq 2,$$

$$y_2 + y_3 \geq 4\,,\;\; y_1 + y_3 \geq 2\,.$$

The solution to this system is given in Figure 7.

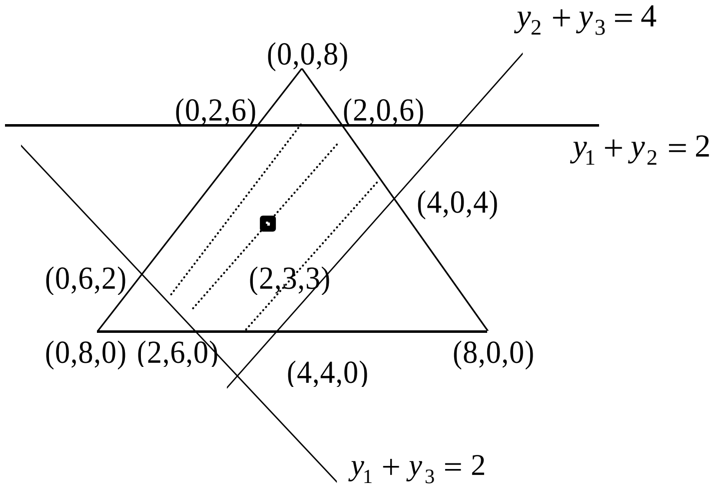

Figure 7. The Core: Case (B).

The most appealing solution to the cooperative game is the center of gravity of the core (2,3,3), which also corresponds to the length of contiguous boundaries between the countries (see Figure 7 (B)).

The next cooperative game in part reflects the mutual location of NATO countries, Slovakia and U

Example 3. The NATO-Slovakia-Ukraine Game: a Simplified Version.

The countries is located as depicted in Figure 8.

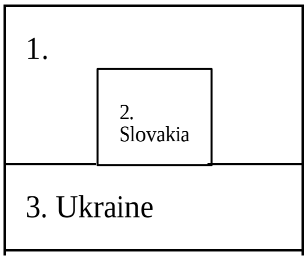

Figure 8. The NATO-Slovakia-Ukraine Game: a Simplified Version.

In this case, $c(\{1\}) = 10$, $c(\{2\}) = 4$, $c(\{3\}) = 8$. In this case, $c(\{1,2\}) = 8$, $c(\{1,3\}) = 16$, $c(\{2,3\}) = 10$, $c(\{1,2,3\}) = 12$. Property (2.1) gives us:

$$x_1 + x_2 + x_3 = 12 \quad x_1 \leq 12,\ x_2 \leq 4,\ x_3 \leq 8,$$

$$x_1 + x_2 \leq 10,\ x_1 + x_3 \leq 16,\ x_2 + x_3 \leq 10.$$

Changing variables ($y_i = c(i) - x_i$),

$$y_1 + y_2 + y_3 = 12,\ y_i \geq 0,\ i = 1,2,3,$$

$$y_1 + y_2 \geq 6,$$

$$y_2 + y_3 \geq 2,\ y_1 + y_3 \geq 4.$$

The core of this cooperative game is given in Figure 9. The center of gravity of the core is the point $(5,4,3)$. That is, entering a coalition with its neighbors is in the interest of each of the countries and the amount of cost savings for the current NATO members are larger than those for Ukraine and Slovakia.

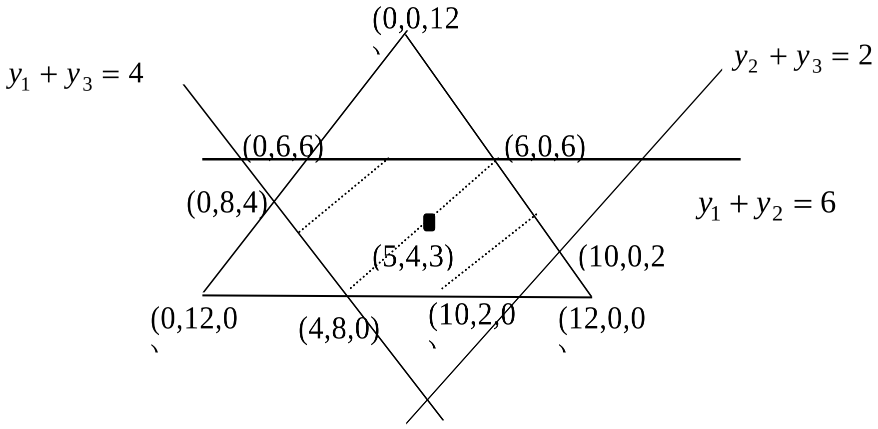

Figure 9. The Core: a Simplified Version of the NATO-Slovakia-Ukraine Game.

An Application to NATO Expansion

The above analysis of cooperative mutual defense games is of help in understanding the current pro

There is no doubt that the most prospective member of the alliance is Slovakia. It has common l

protection. Unfortunately, Romania has lower chances to join NATO, not only due to economic and political causes. Unlike Slovakia, cost savings that would result from Romania being a NATO member would be much lower. Of 2,508 km of its boundaries, Romania has only 443 km of common boundaries with Hungary. From the same angle we can look at the problem of Ukraine's membership in NATO. Of 4,558 km of its land boundaries, Ukraine has common boundaries with two country members of NATO: Poland (428 km), Hungary (103 km), that is, if Ukraine became a member of NATO, there would not be marked cost savings because of a sizable increase in external boundaries' length.

Before considering a more complicated version of the three-person game with NATO, Slovakia, Ul

billion and $35 billion over 10 years; a US Congressional Budget Office (CBO) study, $21 billion to $125 billion, a 1997 NATO study put the figure at $13 billion over 10 years (Behner, 1999). . The ranges suggested by the studies and differences among them to a large extent reflect different ways of allocating costs among members. Generally, expansion cost is subdivided into three parts: new members' costs, current members' costs, and the common NATO infrastructure costs. The 1997 Pentagon estimate saw new members providing $14 billion, current European members providing 12 billion, and the common infrastructure account requiring $9 billion (with only $2 billion of which given by the USA). On the other hand, the NATO study to a great extent excluded costs to current members and set common costs at only 1.5 billion, but kept the cost to new members higher. By adopting this estimate the Pentagon avoided a burden-sharing argument. Note that the new members are not up to accepting high costs concerning upgrading their military to NATO standards (for example, Poland appropriated only a five-year $2.3 billion for this purpose).

In order to take into account the costs of expansion (for example, those related to upgrading the

negative values since according to a number of experts in the long run a country's membership in NATO results in cost savings. That is, the value which z takes on , among other factors, also depends on the length of the period of time over which the model is considered. We also assume that the cost of protecting a unit of a NATO country's boundary is currently equal to 1. Note that today's military spending in Ukraine is without a doubt inadequate and can not ensure the country's security. Its low current level reflects the poor state of the Ukrainian economy, not Ukraine's real defense needs.

In Figure 10, a more complicated version of the NATO-Slovakia-Ukraine game is depicted:

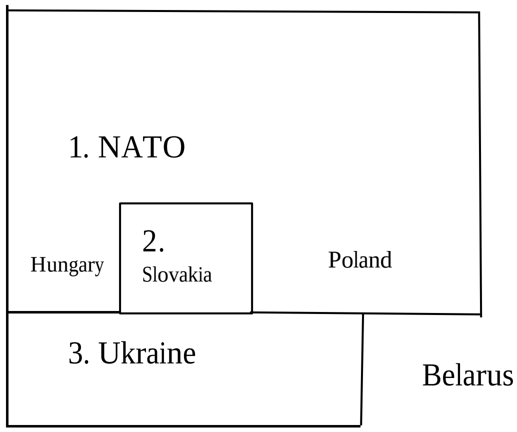

Figure 10. The NATO-Slovakia-Ukraine Game

The costs of protecting the individual players are as follows: $c(\{1\}) = 16$, $c(\{2\}) = 6$, $c(\{3\}) = 12$. As for the costs of protecting the possible coalitions, $c(\{1,2\}) = 14.5 + z$, $c(\{1,3\}) = 23 + 6z$, $c(\{2,3\}) = 10$, $c(\{1,2,3\}) = 18.5 + 5z$.

Property (2.1) gives us:

$$x_1 + x_2 + x_3 = 18.5 + 5z \quad x_1 \le 16,\ x_2 \le 6,\ x_3 \le 12,$$

$$x_1 + x_2 \le 14.5 + z,\ x_1 + x_3 \le 23 + 6z,\ x_2 + x_3 \le 10.$$

Changing variables ($y_i = c(i) - x_i$),

$$y_1 + y_2 + y_3 = 15.5 - 5z,\ y_i \ge 0,\ i = 1,2,3,$$

$$y_1 + y_2 \ge 7.5 - z,$$

$$y_2 + y_3 \ge 8,\ y_1 + y_3 \ge 5 - 6z$$

Note that if z took on extremely high values (that is, if the cost of entering NATO were prohibitive), the core of the game would be empty (there would be no opportunity to create a mutually beneficial coalition). It follows from the following computations: $2(y_1 + y_2 + y_3) = (y_1 + y_2) + (y_2 + y_3) + (y_1 + y_3) \geq 20.5 - 7z$. So if 2(15.5-5z)<20.5-7z, the above system has no solution. That is, if $z > 3.5$, at least one of the players can benefit from not entering the three-person coalition.

Let us graphically determine the core of the game in the short run $(z > 0)$.

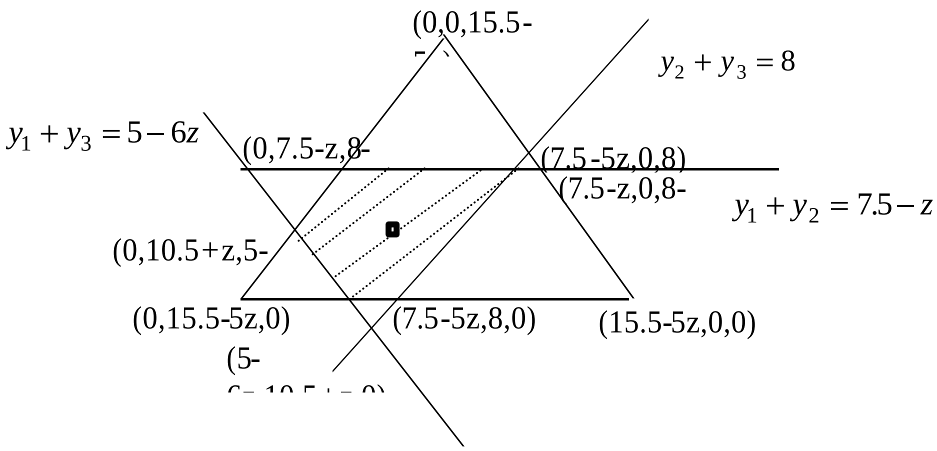

Figure 11. The Core: the NATO-Slovakia-Ukraine Game in the Short Run.

Computing the center of gravity for the core of the game is a tedious problem. At the same time, F

with NATO country members and Ukraine. On the other hand, Ukraine has external boundaries of considerable length with countries not being NATO members and, as a result, Ukraine's burden resulting from entering NATO in the short run will be higher.

At the same time, we have considered NATO countries as a coalition (in essence, one country), a

Obviously, Poland as the interior country gains a large share of cost savings and therefore is more interested in signing a pact than Germany. Similar reasoning allows us to explain why NATO's new eastern members are so much in favor of eastward expansion. They would in this case become interior countries and thus gain essential cost savings. Therefore, we can come to conclusion that if the issue of accepting Ukraine in NATO was on the agenda, the most ardent supporters would be Poland and Hungary.

In the long run $(z = -0.5)$, the core of the game is depicted below.

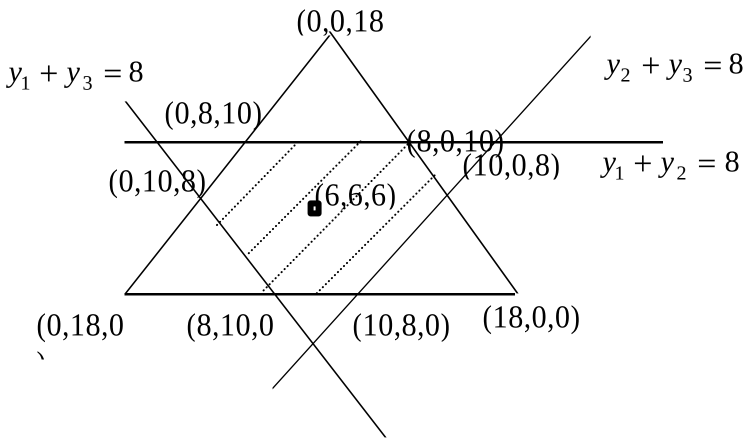

Figure 12. The Core: the NATO-Slovakia-Ukraine Game in the Long Run.

The most appealing solution to the cooperative game is the center of gravity of the core (6,6,6). That is, unlike the short-run case, in the long run the three players benefit equally from the expansion of NATO. This reasoning gives an incentive for NATO to expand eastward and for Ukraine to join NATO, short-run costs notwithstanding.

In most mutual defense games the core is not empty, that is, it is possible for the countries to co games (Perry and Reny, 1994; Evans, 1997) the core coincides with the pure stationary subgame perfect equilibrium payoff set, so studying the problem of existence of a stationary equilibrium in a noncooperative bargaining game is to a considerable extent akin to studying the problem of nonemptiness of its core. On the other hand, in the course of proving the existence of a stationary subgame perfect equilibrium the process of bargaining (the way of coming to the final utility distribution) is often described in an explicit way. The problem of existence of a stationary equilibrium in a bargaining game is studied in the following chapter.

C h a p t e r 3

A NONCOOPERATIVE n -PERSON BARGAINING GAME WITH RANDOM PROPOSERS

The core in a cooperative game represents the feasible set of cooperative opportunities: if the pl considered within the framework of bargaining theory. Important contributions to the well-known programme of "achieving" cooperative game-theoretic concepts via the play of a strategic game was made by Chatterjee *et al*. (1993), Okada (1996), Evans (1997). These authors showed that for a number of coalitional bargaining games any payoff vector in the core of the underlying game is the outcome of some stationary subgame-perfect strategy equilibrium of the bargaining game (without discounting). On the other hand, Chatterjee *et al.* showed that for some strictly convex bargaining games as the discount factor tends to 1, the corresponding sequence of efficient stationary equilibria converges to a point in the core. So the problem of the existence of a stationary strategy equilibrium is akin to the problem of the nonemptiness of the core in this game.

Further we consider the problem of existence of a stationary strategy equilibrium in an n -person ba

The Bargaining Game: Formulation

Let us consider an bargaining game $(N, D, X, \{u_i(\cdot)\})$, where $N = \{1, \ldots, n\}$, $n \geq 2$; the set of de

exists $x \in X$ such that $u_i(x) > 0$, $i = 1, \ldots, n$. Without loss of generality, the interior of X is non-empty (otherwise, we can consider the set as a subset of its affine span).

The process of bargaining can be described in a recursive way as follows:

At round $t = s,\ s \in \{1, 2, \ldots\}$

1. One player is randomly selected as proposer, with player i being selected with probability $1/n$.
2. The proposer makes a proposal $x \in X$.
3. All the other players simultaneously respond by saying "Yes" or "No" to the proposal.
4. If each of them accepts the proposal, the game ends. Otherwise, the process moves to round $s+1$.
5. If $x \in X$ is accepted in period s, player i 's payoff is given by $d^{s-1} u_i(x)$, $d \in (0, 1)$, that is, all players discount their future payoffs by a common discount factor. If no alternative is ever accepted, each player receives a utility of zero.

We assume that every player knows the rules of the game and has perfect information about the

stationary equilibrium is a collection of stationary strategies such that there is no history at which a player benefits from a deviation from his prescribed strategy.

The Central Result

The assumption that players' utility functions are concave is weaker that of strictly concavity of

concave rather than strictly concave.

Proposition. In the above bargaining game, there exists a stationary equilibrium.

Proof. For some $i \in \{1, \ldots, n\}$ and $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X^{n-1}$, let us consider the following

$$\max_{x \in X} u_i(x)$$

s. t. $u_j($

Let us study in detail the set of maximizers of the problem. First of all, show that for any $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X^{n-1}$ the system of inequalities (3.1) is compatible. Consider the following set

$$K_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) =$$
$$= \bigcap_{j \in N-i} \left\{ x \in X : (1-\delta/n) u_j(x) \geq \delta/n \left(\sum_{l=1}^{i-1} u_j(x_l) + \sum_{l=i+1}^{n} u_j(x_l) \right) \right\}$$
$$= \bigcap_{j \in N-i} \left\{ x \in X : \frac{n-d}{d} u_j(x) \geq \left(\sum_{l=1}^{i-1} u_j(x_l) + \sum_{l=i+1}^{n} u_j(x_l) \right) \right\}.$$

Taking into account that $d \in (0,1)$ and $f:[0,1] \to R_+,\ f(d) = \dfrac{n-d}{d}$ is a strictly decreasing function, we come to the conclusion that the following inclusion holds:

$$K_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \supseteq$$

$$M = \bigcap_{j \in N-i} \left\{ x \in X : (n-1)u_j(x) \geq \left(\sum_{l=1}^{i-1} u_j(x_l) + \sum_{l=i+1}^{n} u_j(x_l) \right) \right\}.$$

Since $u_i : X \to R_+, i = 1, \ldots, n$, are concave function, the set M contains the point $x_* = \frac{1}{n-1}\left(\sum_{l=1}^{i-1} x_l + \sum_{l=i+1}^{n} x_l \right)$. Therefore, the set $K_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is a non-empty convex compact set and, as a result, $X_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, the set of elements of X being a solution of the optimization problem $\max_{x \in K_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)} u_i(x)$, is a non-empty convex compact subset of X (the set of maximizers of a concave function on a convex set is a convex set). In order to invoke the Kakutani fixed point theorem, we need to prove that $X_i : X^{n-1} \to 2^X$ is upper semicontinuos. Its semicontinuity is proved in the section "Upper Semicontinuity of Auxiliary Set-Valued Maps."

Further our reasoning is similar to that of Glicksberg (1952, p.173). Let us consider the following s

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} X_1(x_2, x_3, \ldots, x_n) \\ X_2(x_1, x_3, \ldots, x_n) \\ \vdots \\ X_n(x_1, x_2, \ldots, x_{n-1}) \end{pmatrix}.$$

The set X^n is convex compact, values of X_0 are non-empty convex compact subsets of X^n, X_0 is upper semicontinuous on X^n, by virtue of the Kakutani fixed point theorem, there exists a fixed point $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in X^n$ of the map

$(x^* \in X_0(x^*))$.

Let us define the following stationary strategy S_i^* for player i :

A) If player i proposes, he proposes x_i^*;

B) If player i responds to some ongoing proposal x made by player $j \in N - i$, he accepts the proposal if $u_i(x) \geq \frac{d}{n}\left(\sum_{l=1}^{i-1} u_i(x_l^*) + u_i(x) + \sum_{l=i+1}^{n} u_i(x_l^*) \right)$ and rejects it otherwise.

On the basis of the above proved, the strategy profile $(S_1^*, S_2^*, \ldots, S_n^*)$ constitutes a stationary equ

concave utility functions) is a complicated problem lying outside the scope of the thesis. Q.E.D.

Upper Semicontinuity of Auxiliary Set-Valued Maps

In order to show that $X_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) : X^{n-1} \to 2^X$ is upper semicontinuous, let us introd

$u_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \max\limits_{x \in K_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)} u_i(x)$. Then

$$X_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) =$$

$$\{x \in K_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) : u_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = u_i(x)\}$$

In order to prove that $X_i : X^{n-1} \to 2^X$ is upper semicontinuous, it suffices to show that K_i is a continuous set-valued map with compact values (Aubin and Cellina, 1984, Theorem 6, p.53). Note that in this case, the marginal function u_i is continuous as well. Recall that $K_i\colon X^{n-1} \to 2^X$ is determined as follows:

$$K_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = \bigcap_{j\in N-i} \left\{ x \in X\colon \frac{n-d}{d} u_j(x) \geq \Big(\sum_{l=1}^{i-1} u_j(x_l) + \sum_{l=i+1}^{n} u_j(x_l) \Big) \right\}$$

,

The proof of the set-valued map's continuity consists of two steps. Firstly, it is necessary to show that $L_J\colon X^{n-1} \to 2^X$,

$$L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = \left\{ x \in X\colon \frac{n-d}{d} u_j(x) \geq \Big(\sum_{l=1}^{i-1} u_j(x_l) + \sum_{l=i+1}^{n} u_j(x_l) \Big) \right\},$$

$j \in N - i$

is continuous. Since the function u_j is continuous on X, L_j has a closed graph and its values lie in the compact set X, so L_j is upper semicontinuous on X^{n-1} (Aubin and Cellina, 1984, Corollary 1, p. 42). The proof of the lower semicontinuity of L_j on X^{n-1} is more complicated. Let us consider a sequence $\left\{(x_1^m,\ldots,x_{i-1}^m,x_{i+1}^m,\ldots,x_n^m)\right\}_{m=1}^{\infty}$ converging to $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ and choose some $z \in L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. It is necessary to show that there exists a sequence $\left\{z^m\right\}$, $z^m \in L_j(x_1^m,\ldots,x_{i-1}^m,x_{i+1}^m,\ldots,x_n^m)$ converging to z. Consider the sequence $\left\{z^m\right\}$, $\left\|z^m - z\right\| = \min\limits_{y\in L_j(x_1^m,\ldots,x_{i-1}^m,x_{i+1}^m,\ldots,x_n^m)} \left\|y - z\right\|$. Taking into account that $L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ is a convex compact set for each

$(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\in X^{n-1}$, the projection of the point z on the set $L_j(x_1^m,\ldots,x_{i-1}^m,x_{i+1}^m,\ldots,x_n^m)$ is determined in a unique manner (Aubin, 1984, Theorem 2.3). Let us show that $\{z^m\}$, $z^m\in L_j(x_1^m,\ldots,x_{i-1}^m,x_{i+1}^m,\ldots,x_n^m)$ determined in the above way converges to z. By contradiction, assume that there is a subsequence of $\{z^m\}$ converging to $z^*\neq z$. In this case, for simplicity of our notation, without loss of generality, we can assume that all the sequence $\{z^m\}$ converges to $z^*\neq z$. Since L_j is upper semicontinuous on X, $z^*\in L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. The fact that the sequence $\{z^m\}$ converges to z^* implies that there exists $\varepsilon>0$ such that $z^m\notin B_\varepsilon^k(z)$, $m=1,2,\ldots$, where $B_\varepsilon^k(z)=\{y\in R^k:\|y-z\|\leq\varepsilon\}$. Then

$$B_\varepsilon^k(z)\cap L_j(x_1^m,\ldots,x_{i-1}^m,x_{i+1}^m,\ldots,x_n^m)=\emptyset,\ m=1,2,\ldots.$$

Let us show that in this case the point z is a local maximizer of u_j on $B_\varepsilon^k(z)\cap X$, that is, $z\in\arg\max\limits_{x\in B_\varepsilon^k(z)\cap X}u_j(x)$. By contradiction, assume that there exists $z_1\in B_\varepsilon^k(z)\cap X$, $z_1\neq z$ such that $u_j(z_1)>u_j(z)$. Taking into account that

$$\frac{n-d}{d}u_j(z)<\left(\sum_{l=1}^{i-1}u_j(x_l^m)+\sum_{l=i+1}^{n}u_j(x_l^m)\right),\ \frac{n-d}{d}u_j(z)\geq\left(\sum_{l=1}^{i-1}u_j(x_l)+\sum_{l=i+1}^{n}u_j(x_l)\right),$$

we come to the conclusion that

$$\frac{n-d}{d}u_j(z)=(\sum_{l=1}^{i-1}u_j(x_l)+\sum_{l=i+1}^{n}u_j(x_l))<\frac{n-d}{d}u_j(z_1)$$

and beginning with some number N_1 the following inequality holds

$$\frac{n-d}{d}u_j(z_1)>(\sum_{l=1}^{i-1}u_j(x_l^m)+\sum_{l=i+1}^{n}u_j(x_l^m)),\ m=N_1,N_1+1,\ldots\ .$$

Therefore, $z_1\in B_e^k(z)\cap L_j(x_1^m,\ldots,x_{i-1}^m,x_{i+1}^m,\ldots,x_n^m)$, which contradicts (3.2).

Having proved that z is a local maximizer of u_j on $B_e^k(z)\cap X$, by invoking the fact that the function u_j is concave on X, we can conclude that z is a global maximizer of u_j on X. So $z\in L_j(x_1^m,\ldots,x_{i-1}^m,x_{i+1}^m,\ldots,x_n^m)$, $m=1,2,\ldots$, which contradicts to (3.2). The obtained contradiction proves that the above sequence $\{z^m\}$ converges to z and the lower semicontinuity of L_j at the point $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\in X^{n-1}$.

Let us come back to proving the continuity of K_i on X^{n-1}. The map is upper semicontinuous on

$$K_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)=\bigcap_{j\in N-i}L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n).$$

The simplest general condition guaranteeing that the intersection of two lower semicontinuous convex-valued maps F and G acting from a metric space V to R^m is lower semicontinuous in $v_0\in V$ can be written as follows (Aubin and

Frankowska, 1990, Proposition 1.5.1, p. 49,; Chikrii and Prokopovich, 1995, Theorem 1, p. 99):

$$0 \in \operatorname{int}(F(v_0) - G(v_0)),$$

where $\operatorname{int} A$ denotes the interior of the set A.

In particular, in order to prove lower semicontinuity of K_i on X^{n-1}, it is enough to show that for any $(x_1,\ldots x_{i-1},x_{i+1},\ldots,x_n)$ there exists

$z \in \operatorname{int} L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$, $j = 1,\ldots,i-1,i+1,\ldots,n$

As it was noted above, the point $x_* = \dfrac{1}{n-1}\left(\sum_{l=1}^{i-1} x_l + \sum_{l=i+1}^{n} x_l\right)$ belongs to $L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,$ that u_j is a continuous function and $f:[0,1] \to R_+$, $f(d) = \dfrac{n-d}{d}$ is a strictly decreasing function, we come to the conclusion that there exists $e_j > 0$ such that $B^k_{e_j}(x_*) \cap X \subset L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. Let $e^* = \min\{e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n\}$. Then $B^k_{e^*}(x_*) \cap X \subset L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$, $j = 1,\ldots,i-1,i+1,\ldots,n$. Note that $\operatorname{int}(B^k_{e^*}(x_*) \cap X) \neq \varnothing$ and $\operatorname{int}(B^k_{e^*}(x_*) \cap X) \subset \operatorname{int} L_j(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$, $j = 1,\ldots,i-1,i+1,\ldots,n$. Therefore, a z such that inclusions (3.3) hold exists and K_i is lower semicontinuous at the point $(x_1,\ldots x_{i-1},x_{i+1},\ldots,x_n) \in X^{n-1}$. Since K_i is both lower and upper semicontinuous on X^{n-1}, we conclude that K_i is continuous on X^{n-1} and $X_i : X^{n-1} \to 2^X$ is an upper semicontinuous set-valued map with non-empty convex compact values.

Chapter 4

CONCLUSIONS

NATO expansion is currently the centerpiece of U.S. security policy toward Europe. Of late, the Alliance has reaffirmed that the door to new members remains open and enhanced practical military cooperation and political dialogue with those countries who seek membership. Ukraine as a partner in the Partnership for Peace (PfP) program has developed a new security relationship with the Alliance members and other PfP partners. Moreover, NATO and Ukraine have strengthened their distinctive partnership recently. But costs of expansion of NATO are large. New NATO members would have to devote enormous funds to buy modern weapons and communication systems compatible with those used by the Western countries (an increase of 60 to 80 percent over current military expenditures). At the same time, alliance formation depends on whether prospective members consider their membership as providing net gains. In Chapter 2 of this work, we analyze the cost and benefits to Ukraine connected with joining NATO. On the basis of applying Gardner's idea (1995) concerning studying mutual defense games within the cooperative game theory framework, the benefits to Ukraine from forming an alliance with a number of her neighbor countries are investigated. Studying the auxiliary cooperative mutual defense games is of theoretical interest on its own.

After that, some relevant issues of NATO expansion are raised and explored. A "NATO-Slovakia-Ukraine" model is proposed and the set of its cooperative solutions is determined. It is shown that although in the short run the burden of the NATO expansion is divided unequally among the three players, the long-run cost savings for them from the NATO expansion will be to a considerable extent equalized.

Owing to the close connection between the core of a coalitional game and the set of stationary subgame perfect equilibrium payoffs of the corresponding bargaining game, the problem of nonemptiness of the core can often be interpreted as the problem of existence of a stationary subgame perfect equilibrium in the corresponding bargaining game. In Chapter 3, the problem of existence of a stationary equilibrium in a unanimous bargaining game with the random selection of proposers is studied. In this game, the conventional assumption of the strict concavity of players' utility functions is replaced with that of their concavity. The proof of the existence of a stationary equilibrium is based on invoking the Kakutani fixed point theorem. In order to make it possible, upper semicontinuity of a number of auxiliary set-valued maps is investigated. Thus, within the framework of the original approach to formalizing n-person bargaining proposed by the author, invoking the Kakutani fixed point theorem is thoroughly substantiated.

MATHEMATICAL APPENDIX

We give in this section a number of definitions and assertions which are used in Chapter 3. In what follows X and Y Hausdorff topological spaces. Let F be a set-valued map with non-empty values.

Definition 1. We say that F is upper semicontinuos at $x^0 \in X$ if for any open neighborhood N containing $F(x^0)$ there exists a neighborhood M of x^0 such that $F(M) \subset N$.

We say that F is upper semicontinuous if it is so at every $x^0 \in X$.

Proposition 1 (Aubin and Cellina, 1984, Corollary 1, p. 42)**.** Let G be a set-valued map from X to a compact space Y whose graph is closed. Then G is upper semicontinuous.

Proposition 2 (Aubin and Cellina, 1984, Theorem 1, p. 41). Let F and G be two set-valued maps from X to Y such that, $\forall x \in X, F(x) \cap G(x) \neq \varnothing$. We suppose that:

i) F is upper semicontinuous at x_0,

ii) $F(x_0)$ is compact,

iii) the graph of G is closed.

Then the set-valued map $F \cap G : x \to F(x) \cap G(x)$ is upper semicontinuous at

x_0 .

Definition 2. We say that F is lower semicontinuous at $x^0 \in X$ if for any $y^0 \in F(x^0)$ and any neighborhood $N(y^0)$ of y^0, there exists a neighborhood $N(x^0)$ of x^0 such that

$$\forall x \in N(x^0),\ \ F(x) \cap N(y^0) \neq \varnothing .$$

We say that F is lower semicontinuous if it is lower semicontinuous at every $x^0 \in X$.

The above definition could be phrased as follows: given any generalized sequence x_m converging to x^0 and any $y^0 \in F(x^0)$, there exists a generalized sequence $y_m \in F(x_m)$ that converges to y^0. When X and Y are metric, this last characterization holds true with usual (i.e., countable) sequences.

Definition 3. A set valued map F from X to Y is said to be continuous at $x_0 \in X$ if it is both upper semicontinuous and lower semicontinuous at x^0. It is said to be continuous if it is continuous at every point $x \in X$.

Let G be a set-valued map from Y to X and W be a real-valued function defined on $X \times Y$. We consider the family of maximization problems

$$V(y) = \sup_{x \in G(y)} W(x, y) ,$$

which depend upon the parameter y. The function V is called the marginal function and the set-valued map M associating to the parameter $y \in Y$ the set

$$M(y) = \{x \in G(y) : V(y) = W(x,y)\}$$

of solutions to the maximization problem $V(y)$ is called the marginal map.

Proposition 3 (Aubin and Cellina, 1984, Theorem 6, p.53). Suppose that

i) W is continuous on $X \times Y$,

ii) G is continuous with compact values.

Then the marginal function V is continuous and the marginal set-valued map M is upper semicontinuous.

Proposition 4 (Aubin and Frankowska, 1990, Proposition 1.5.1, p.49). Consider a metric space X, two normed spaces Y and Z, two set-valued maps G and F from X to Y and Z respectively, and a single-valued map f from $X \times Z$ to Y satisfying the following assumptions:

i) G and F are lower semicontinuous with convex values;

ii) f is continuous;

iii) $\forall x \in X$, $u \mapsto f(x,u)$ is affine.

We posit the following condition:

$\forall x \in X$, $\exists g > 0,\ d > 0,\ c > 0,\ r > 0$ such that $\forall x' \in B(x,d)$ we have

$$g\ B \subset f(x', F(x') \cap rB_Z) - G(x').$$

Then the set-valued map $R: X \to 2^Z$ defined by

$$R(x) = \{u \in F(x) \colon f(x,u) \in G(x)\}$$

is lower semicontinuous with nonempty convex values.

Proposition 5 (Chikrii and Prokopovich, 1995, Theorem 1, p. 99). Let us consider a metric space V and set-valued maps F and G from V to R^k with convex values such that

i) F and G is lower semicontinuous at $v_0 \in V$;

ii) $0 \in \operatorname{int}(F(v_0) - G(v_0))$.

Then the set-valued map $C\colon V \to 2^{R^k}$, $C(v) = F(v) \cap G(v)$ is lower semicontinuous at v_0.

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