EFFICIENCY OF A GROUP AND REALLOCATION OF RESOURCES

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Abstract

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In real life, as well as in theory, the goals of different units within an enterprise may be inconsistent with the eventual goal of an enterprise as a whole. In this work, we hypothesize that efficient operation of constituent units does not necessarily imply efficiency on the enterprise level. We show that, under certain assumptions on technology, it is possible to identify theoretically, and estimate empirically, the extent to which the performance of an enterprise as a group of units can be enhanced, even if all units are individually efficient. The existence of such potential improvement is attributed to non-optimal allocation of resources across the units, from the point of view of an enterprise. We merge the theoretical findings of Li and Ng (1995) and Färe and Zelenyuk (2003), and come up with the appropriate group-wise efficiency measures, which allow for the possibility of resources reallocation. By means of hypothetical example, we demonstrate that pure output and revenue gains from reallocation of resources within the group of units may be, indeed, substantial. In our example they amount to about 31% of extra revenue.

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Chapter 1

INTRODUCTION

"…two firms which, taken individually, are perfectly efficient
… are not perfectly efficient when taken together."
Farrell (1957), *Journal of the Royal Statistical Society*, p.261

In the economic theory, firms are oftentimes viewed as single units, which state their goals and do their best, so as to achieve these goals at minimum cost. Implicitly assumed is that the management (*i.e.*, headquarters) has full control over the performance of the units within the organization. However, this assumption is unlikely to be true in real life. Presence of agency and information problems in such a setup has motivated researchers to turn to so called "micro-micro-level" of analysis, so as to study the behavior of different units *within* the enterprise. Due to the agency problem and imperfect information, these units may behave not optimally from the point of view of the headquarters (and shareholders), and creation of "right" incentives is often very complicated and can not be prescribed as an one-for-all cure. Particularly, as argued by Leibenstein (1966) in his development of X-inefficiency theory, unobserved suboptimal decisions may result in productive inefficiency. For example, the manager of a bank's branch is often concerned with the higher profits of the branch, rather then with those of the bank as a whole. This line of thought can be continued to the level of divisions and individual employees as well. Being motivated mainly by the fact that the goals, in terms of output, revenues or profits, of different units within the enterprise may not be consistent with the eventual goal of the enterprise as a whole, we formulate the hypothesis that *efficient operation of constituent units does not imply efficiency on the enterprise level.*

The main result of this work is that, under certain assumptions on technology, we are able to identify theoretically, and estimate empirically, the extent to which enterprise efficiency can be enhanced, even if all its individual units operate efficiently. The source of such inefficiency is attributed to *non-optimal allocation of inputs (thus, activity) across the units*, from the point of view of the enterprise. In this work, we develop the measure of such inefficiency. It is named *"reallocative efficiency"* and it indicates *potential* revenue and output gains solely due to *reallocation* of inputs across the units of an enterprise. Specifically, we formulate a theorem, which states that *group efficiency when inputs can be reallocated across the units is equal to group efficiency when such reallocation is not possible, adjusted by group reallocative efficiency.*

However, in order to understand the essence of this result, we should first introduce the reader to a problem of estimating efficiency of the group of units in the light of possibility of inputs reallocation. We emphasize, as well as some other researchers did (see the discussion of the prior work below), that the measurement of the efficiency of a group should depend on whether reallocation of inputs across the units is possible or not. If it is not possible, one may aggregate individual efficiency scores to get the efficiency of the whole system. For example, a weighted average measure may be used, as it was suggested by Färe and Zelenyuk (2003). We follow the Farrell's tradition (Farrell, 1957) and name the weighted average measure of group efficiency as a *"structural efficiency of the group"*[1]. Note that if structural efficiency is used, efficient operation of all units would necessarily imply efficiency on the group level. This result follows from the theorem, formulated and proved in Färe and Zelenyuk (2003), which states that if all units face the same output prices, group maximum revenue [function] is equal to the sum of individual maximum revenues. This result comes from the notion of group output set, which is defined by the authors as the sum of individual output sets. However, if inputs can be reallocated within the group, we can not define group production possibilities as simply the sum of individual ones. In short, in this work we redefine the set of production possibilities of the group and prove that it is less restrictive than the sum of individual ones. This led us to the new notion of group *"potential revenue function"*, which is proved to be higher than or equal to the sum of individual maximum revenues. Accordingly, we propose a new measure of revenue efficiency of the group, so called *"potential revenue efficiency"*, which accounts for the possibility of inputs reallocation across the units. It should be noted that similar ideas were put forward by other scholars, for example by Li and Ng (1995), Bogetoft and Wang (1999), Coelli *et. al.* (2003), *etc.* In this work, we extend their results further in four major aspects. In what follows, we describe each portion of our contribution in more details, moving from the most important one to the relatively minor extensions.

First, *we decompose overall potential efficiency of the group, i.e. potential revenue efficiency, into two parts: structural revenue efficiency and revenue reallocative efficiency* (this is the theorem we have already mentioned above). The first component is the weighted average measure, defined by Färe and Zelenyuk (2003), and it presumes impossibility of inputs reallocation. The second component is defined in this work. It indicates revenue gains due to the reallocation of inputs across the units and serves as a link between the two group measures, potential efficiency and structural efficiency.

The second extension to the prior works is *the decomposition of the potential revenue efficiency, into technical and allocative components: "potential technical efficiency" and "potential allocative efficiency"*. The first component, potential technical efficiency, indicates unrealized output gains of the group of units, including the output gains from

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[1] Which is the generalization of Farrell's "structural efficiency of the industry". Below we use the term "structural efficiency" and the term "weighted average measure" interchangeably.

reallocation of inputs across them. The second component, potential allocative efficiency, reflects group revenue gains due to the switch to the product mix (and hence, input mix), which maximizes *potential* group revenue.

Third, *we decompose revenue reallocative efficiency (the measure of gains from reallocation of inputs across the units) according to its sources, into technical and allocative components*. The first one, "technical reallocative efficiency", represents potential *output gains* solely due to reallocation. It reflects the extent, to which total enterprise's output can be increased further over the aggregate output of technically efficient units. The second source of revenue gains is "allocative reallocative (in)efficiency". Despite its somewhat confusing name, it has an important meaning. Since technical efficiency of the enterprise does not imply revenue efficiency, which is conditioned on output prices, it may be the case that enterprise's output mix (and thus, input mix) is not optimal for its revenue maximization. Moreover, the agency problem may result in the situation, when individual units prefer suboptimal, from the enterprise view, output mix. Allocative reallocative inefficiency reflects a *mismatch* between the product mix, which maximizes the sum of individual revenues (*i.e.*, group maximal revenue when inputs reallocation is not possible), and the product mix maximizing group *potential* revenue (*i.e.*, maximal group revenue when reallocation is possible).

Finally, we demonstrate on the example *how, in practice, one may determine from and to which units reallocation may be necessary,* to achieve higher group revenues.

Although the idea of reallocative efficiency is not new and, in fact, has its roots in the seminal paper of Farrell (1957) (see the epigraph), researchers tended to overlook it. When there was a need for group-wise efficiency estimation, simple or weighted average of individual scores were used, with no theoretical grounds for such aggregation. Though, these were provided in the work of Färe and

Zelenyuk (2003), where they suggest the use of weighted average, with the weights derived from economic optimization, and being either the revenue shares (for output-oriented case), or price independent weights. We stress that such aggregation provides the researcher with the full picture of group inefficiency only in case inputs reallocation is not possible within this group – which is consistent with the context of independently operating units. However, if one wants to consider the efficiency of a group, within which the reallocation of inputs is possible (as in the example of a large bank with many branches) then the measures of *potential* efficiency should be used. If, instead, in such a situation the measure of structural efficiency is employed, the efficiency of the group will be, in general, underestimated.

In order to illustrate how efficiency of the enterprise as a group of multiple units behave, before and after reallocation, we provide a hypothetical example. In this example, we demonstrate that there can be, indeed, substantial output and revenue gains, which become available just due to reallocation of inputs across the units. For instance, in our example pure revenue gains from reallocation amounts to about 31%. By simulating total and partial (only across some units) reallocations, we show *how* these gains can be seized.

The paper is organized as follows. First, we present the existent theoretical background for the group efficiency estimation, while assuming that each unit operates independently in sense that inputs can not be reallocated across the units. Then, we introduce the possibility of inputs reallocation. We discuss the prior work related to the group efficiency measurement with the possibility of inputs reallocation and provide theoretical extensions of the existent studies. We proceed with the estimation methodology, which is accompanied with the empirical example. Finally, we discuss the limitations and possible extensions to our work and conclude.

C h a p t e r 2

LITERATURE REVIEW

Our analysis is not *ad hoc* and is based on the neo-classical economic theory. In particular, we use Shephard's (1970) axiomatic approach to production theory, related to it Debreu (1952)/Farrell (1957) notion of efficiency, which was extended later, in particular by Charnes, Cooper and Rhodes (1978). We rely upon recent discoveries in the field of aggregation by Li and Ng (1995) and Färe and Zelenyuk (2003), as well as other related works. Similar ideas may be also found in Bogetoft and Wang (1999), who study potential gains from mergers. Briec *et al.* (2003) applies the idea of group's potential efficiency to the aggregation of directional distance functions. Färe *et al.* (1994) propose the measure of the "efficiency due to product diversification", which is conceptually the same to the measure of gains due to reallocation of inputs. Yet, it is based on the comparison of group potential minimum costs to the sum of individual cost functions, while reallocative efficiency compares respective revenue functions. Coelli *et al.* (2003) make an attempt to develop a measure of potential and reallocative efficiency, but their work seem to be incomplete, in a sense that it does not discover a full range of relationships between structural and potential measure, and sometimes it lacks theoretical justification[2]. In what follows, we discuss the aforementioned works, as well as the other research on measurement the efficiency of the group of decision making units, in more details.

[2] Their two major propositions (see Coelli et al. (2003), (4.5) and (4.13)), which were central to the establishing the relation between potential and structural efficiency measures, were not actually proved.

In his seminal work, Farrell (1957) suggested the weighted measure of industry efficiency, which he named *structural efficiency*, since it was supposed to provide the information on relative distance of the whole industry to the frontier, or, alternatively, "the extent to which an industry keeps up with the performance of its own constituent firms" (Farrell (1957), p.262)[3]. However, he did not elaborated on that issue much, leaving it for further research. Since then, as noted by Førsund and Hjalmarsson (1979), the majority of studies used this structural efficiency measure to infer on industry performance as a whole. Yet, the theoretical framework for such aggregation had not been rigorously developed until the work of Färe and Zelenyuk (2003), where necessary theoretical grounds were provided. Over the time, some authors also elaborated on aggregation issues. So, Blackorby and Russell (1999) derived the conditions for consistent aggregation of efficiency measures and concluded that they are "quite stringent" (such as linear technology). Later, Li and Ng (1995) successfully overcame the problems pointed by Blackorby and Russell by introducing shadow prices as weights.

However, some other ideas regarding group efficiency measurement appeared over this time period. One of them was the proposition put forward by Førsund and Hjalmarsson (1979). They suggested using technical efficiency of the "average firm" in the group as an estimate of industry efficiency[4], instead of the weighted average of estimated individual scores initially proposed by Farrell (1957). The "average firm" is just the firm producing group average output from average inputs. Unfortunately, they did not supported their guess with theoretical grounds. So, either they did not realize that using the average firm as a benchmark implicitly assumes possibility of altering allocation of inputs

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3 Hereafter, note that the notion of "industry efficiency" is equivalent to the notion of "efficiency of the group of units". Regardless whether the group is an industry or a multi-unit enterprise.

4 They look only at technical efficiency, so more accurate is to say "technical structural efficiency".

endowments across the firms, or just considered this too trivial to be mentioned explicitly.

Here it is necessary to note, that in fact it was Farrell (1957), who was the first to mention the idea of the average firm. But it was perhaps too briefly outlined to inspire further development by others. He wrote: "It might be thought … that the technical efficiency of an industry … would be simply weighted average of technical efficiencies … of constituent firms. This is basically true, but it needs to be qualified, for in so far as its constituent firms use their inputs in different proportions, this dispersion will reduce the technical efficiency of the industry. This can easily be seen from the fact that the average of two points on [isoquant] will in general lie beyond [isoquant] – that is, that two firms which, taken individually, are perfectly efficient … are not perfectly efficient when taken together". And concluded on the same page: "Thus, with respect to a given efficient isoquant, the technical efficiency of an industry will tend to be somewhat less than the weighted average of the technical efficiencies of its constituent firms" (Farrell (1957), p.261).

This was exactly the result obtained empirically by Førsund and Hjalmarsson (1979) [5]. Later, Li and Ng (1995) justified it theoretically. Present work was much influenced by the paper of Li and Ng (1995), in which they address the problem of input reallocation directly. In their work the idea of reallocation was theoretically connected to the average-firm measure. Specifically, they proved that [technical] efficiency of the group of firms, when reallocation of inputs across them is possible, is equal (under certain assumptions on the technology) to the [technical] efficiency of the average firm. In addition, they provided decomposition of group technical efficiency into technical, allocative and reallocative efficiencies. However, they did not distinguish between technical reallocative efficiency (which deals with aggregate output maximization) and revenue reallocative efficiency (which concerns aggregate revenue maximization). So, their decomposition seems to lack for an applicability, which goes from the way the authors define reallocative efficiency. It may be also seen from the fact that they had allocative efficiency term in the decomposition of group technical efficiency, while the logic suggests that the latter should not depend on the output mix. In this work we make an attempt to overcome these problems.

Nevertheless, there was the critique of the average firm approach. Particularly, Ylvinger (2000) stressed that the average firm's technical efficiency do not actually measure *industry* technical efficiency. He criticized heavily the work of Førsund and Hjalmarsson (1979) and argued that they obtained very misleading results. Indeed, this is true, since group efficiency, when estimated using the average firm, consists of two parts: pure technical efficiency and technical reallocative efficiency (see Li and Ng (1995), and the theoretical section of the present work). The former component indicates how far group actual output is from the aggregate output when all firms in the group are technically efficient, the latter – how group output may be increased further to achieve potential group output, if inputs are reallocated. So if these two parts are not separated, it may lead to the problem with interpretation: one group may be inefficient because its constituent firms are not, the other – because of large reallocative inefficiency. However, Ylvinger explicitly assumed that inputs reallocation was not possible. He wrote: "The average-unit [measures] should thus clearly not be used to evaluate technical efficiency at the industry level when reallocation of inputs across firms is not allowed" (Ylvinger (2000), p.167). But in case of no

[5] They estimated industry efficiency as a weighted average and as the efficiency of the average firm and noted that the latter indeed always exceeded the former. However, they did not even briefly mentioned Farrell's ideas, as well as they did not connected this observation with the issue of inputs reallocation.

reallocation the usage of average firm as a benchmark in fact does not have theoretical grounds. So, first he did not notice that Førsund and Hjalmarsson actually had not assumed no reallocation (neither possibility of reallocation), and second, he did not make any suggestion for the case of possible reallocation. Ylvinger also pointed out that there were "no consensus between the results from the average-unit measure and the [weighted] average of individual efficiencies…" (Ylvinger (2000), p.168). The present work contributes to the achievement of such a consensus. Once again, despite some misunderstandings, Ylvinger was definitely right that the reporting of aggregate efficiency based on the average firm's efficiency alone would lead to misleading, and perhaps harmful conclusions, which again motivates paying attention to the measure of reallocative efficiency.

It should be noted that Färe *et al.* (1994) in their book "Production frontiers" also presented a notion of reallocation, though it was made for the input-oriented case. That is, their measure estimates "potential economies in input due to output reallocation" and they called it "efficiency due to product diversification" (Färe *et al.* (1994), p.263). It is another interesting direction for study, since the idea seems to be similar to the idea of gains from inputs reallocation.

Recently, a paper by Coelli *et al.* (2003) appeared. They base their theoretical part on Li and Ng (1995) results, but employ a more relevant definition of [revenue] reallocative efficiency, which is the same as we use in this work. However, they do not decompose the group potential revenue efficiency measure into technical and allocative terms. They also do not provide the decompositions of potential technical and allocative efficiencies into structural measures, which do not allow for reallocation, and reallocative efficiency measures. In other words, they disregard possible output gains solely from reallocation, as well as the gains from switching to group-wise optimal output mix. In this work we successfully address these issues. Nevertheless, particular advantage of their work is that they construct a dual measure to [group] reallocative efficiency – the measure of efficiency due to reallocation of outputs, which is just the same as "efficiency due to product diversification" proposed by Färe *et al.* (1994), which we have already mentioned above.

Very interesting results were obtained by Bogetoft and Wang (1999), in their study of efficiency gains from merging firms. Though it was not stated explicitly, they actually defined the measure of technical reallocative efficiency (the measure of gains from reallocating inputs across the units), but they did it in a different way that we (as well as Li and Ng (1995), and the others) do. They proposed first to adjust individual outputs by individual technical efficiency, and then to estimate potential technical efficiency of the group. In fact, this should give the output gains solely due to reallocation of inputs. However, it seems that they were not acquainted with the results of the abovementioned works on the issue of estimating group efficiency, and thus they did not connected their ideas with the prior developments. Nevertheless, their measure has an appealing advantage of not depending on prices information, in contrast to the ours. At this stage, we leave the question of theoretical soundness of their measure for further investigation.

It is worth to mention that the work of Färe and Zelenyuk (2003) avoids the issue of resources reallocation simply because of their definition of aggregate production possibilities set. They define it as the sum of individual output sets, so input endowments are constrained, while the aggregate production set, which allows for reallocation of inputs, should be defined as a set of aggregate outputs produced from aggregate inputs. Once the reallocation is allowed, the definition of group technology should be reconsidered and the notion of reallocative efficiency should be introduced. An this is exactly what we do in this work. In fact, we connect the results on measuring group efficiency obtained by Färe and Zelenyuk (2003) and those of Li and Ng (1995), where the measures of potential and reallocative efficiency are introduced.

The above discussion of previous works on the issues of measurement aggregate efficiency of the group of decision making units shows definite gaps in this field and, in some cases, inconsistencies and even misunderstandings. This work is expected to fill those gaps and put the ideas in order, as well as to stress once again the importance of the problem of optimal allocation of economic resources.

C h a p t e r 3

THEORETICAL FRAMEWORK

3.1. INDIVIDUAL TECHNOLOGY AND EFFICIENCY MEASURES

We begin with the assumptions on individual technology, which are known as Axioms of Technology Characterization, or simply, Regularity Conditions (see Färe and Primont (1995)):

Axiom 1. "No free lunch": $y \notin P(0_N), \forall y \geq 0_M$

Axiom 2. "Producing nothing is possible": $0_M \in P(x), \forall x \in \Re_+^N$

Axiom 3. "'Boundness' of the Output set": $P(x)\, is\, a\, bounded\, set, \forall x \in \Re_+^N$

Axiom 4. "'Closedness' of the Technology set T": $Technology\ set\ T\ is\ a\ closed\ set$

Axiom 5. "Free disposability of Outputs": $y^0 \in P(x) \Rightarrow y \in P(x), \forall y \leq y^0, x \in \Re_+^N$

We assume there are $k = 1,...,K$ units $(K > 1)$ within the enterprise (group), which produce outputs $y^k \equiv (y_1^k,...,y_M^k)' \in \Re_+^M$ from inputs $x^k \equiv (x_1^k,...,x_N^k)' \in \Re_+^N$, according to the *individual technology*:

$$T^k \equiv \{ (x^k, y^k) : \ "y^k\, can\, be\, produced\, from\, x^k" \}. \qquad (3.1.1)$$

Equivalently, technology of each unit k can be characterized by its *output set*:

$$P^k(x^k) \equiv \{\, y^k : \ (x^k, y^k) \in T^k \} . \tag{3.1.2}$$

Individual revenue function is defined as the maximum revenue achievable given production technology, inputs endowment and output prices:

$$R^k(x^k, p) \equiv \max_y \{\, py : \quad y \in P^k(x^k) \} , \tag{3.1.3}$$

where $p \equiv (p_1, ..., p_M) \in \Re_+^M$ is a price vector, *common to all firms*. That is, prices are assumed to be the same for all units – a necessary assumption for aggregation of revenues of different units, which we will use further (for details, refer to Färe and Zelenyuk (2003)). Note that $R^k(x^k, p)$ represents a dual characterization of $P^k(x^k)$, as was shown by Shephard (1970).

Individual revenue efficiency is defined as the ratio of individual maximum revenue to its observed revenue:

$$RE^k \equiv RE^k(x^k, y^k, p) \equiv \frac{R^k(x^k, p)}{py^k} . \tag{3.1.4}$$

Individual technical efficiency of *k*-th unit is defined, following Farrell (1957), as a scalar measure of maximum possible radial expansion of output vector within the individual output set[6]:

$$TE^k \equiv TE^k(x^k, y^k) \equiv \max_\theta \{\theta > 0 : \theta y^k \in P^k(x^k)\} . \tag{3.1.5}$$

[6] Hereafter, we do not explicitly include the word "output" in the terms used. Since only output-oriented case is considered in this work, thus distinction with input-oriented measures is unnecessary. The development of the input-oriented case would be similar and is omitted for the sake of brevity.

Accordingly, *individual technically efficient output* is equal to the actual observed output multiplied by individual output technical efficiency score, $\overset{*}{y}^{k} = y^{k} \cdot TE^{k}$. Finally, we define *individual output allocative efficiency* as the ratio of individual maximal revenue to the revenue obtained from individual technically efficient output:

$$AE^{k} \equiv AE^{k}(x^{k}, y^{k}, p) \equiv \frac{R^{k}(x^{k}, p)}{p\overset{*}{y}^{k}}. \qquad (3.1.6)$$

The intuition behind this latter measure is that technically efficient output does not guarantee the highest possible revenues, just because of sub-optimal output mix. AE^{k} reflects the discrepancy between individual maximal revenue and the revenue obtained from individual technically efficient output. Note that, since $R^{k}(x^{k}, p)$ is a solution to the maximization problem, the inequality $R^{k}(x^{k}, p) \geq p\overset{*}{y}^{k}$ (also known as Mahler inequality) must necessarily hold and hence $AE^{k} \geq 1$.

Using the expressions (3.1.4) and (3.1.6), we obtain the famous decomposition of revenue efficiency into technical and allocative components, which will be useful for further treatment:

$$RE^{k} = TE^{k} \cdot AE^{k}. \qquad (3.1.7)$$

3.2. GROUP EFFICIENCY MEASUREMENT WHEN INPUTS CAN NOT BE REALLOCATED

In this section we turn to the *group* efficiency measures, following Färe and Zelenyuk (2003). In the other sections of the paper, we sometimes call them *structural efficiency* measures, in order to emphasize that the allocation of total inputs endowment is taken as given. Here the use is made of the definition of *group technology*, which is characterized through its output set. We begin with the notion of *group output set*, which is defined as the sum of individual output sets:

$$\bar{P}(x^1,...,x^K) \equiv \sum_{k=1}^{K} P^k(x^k) \,. \tag{3.2.1}$$

Group revenue function can be defined on $\bar{P}(x^1,...,x^K)$, as the maximal revenue which can be obtained from the group production at the price p, when all units produce from their input endowments:

$$\bar{R}(x^1,..,x^K,p) \equiv \max_y \{\, py : \qquad y \in \bar{P}(x^1,..,x^K)\} \,, \tag{3.2.2}$$

which was proved to be equal to the sum of individual revenue functions:

Lemma 1. If technology satisfies Regularity Axioms 1-5, the following equality holds:

$$\bar{R}(x^1,..,x^K,p) = \sum_{k=1}^{K} R(x^k,p) \,. \tag{3.2.3}$$

Proof: refer to Färe and Zelenyuk (2003), p. 620.

Parallel to the individual case, *group revenue efficiency* is defined as the ratio of group maximal revenue to the observed total revenue of the group of units:

$$\overline{RE} \equiv \overline{RE}(x^1,..,x^K,Y,p) \equiv \frac{\bar{R}(x^1,..,x^K,p)}{pY} \,. \tag{3.2.4}$$

It was shown, using the result (3.2.3), that the latter is equal to the weighted sum of individual revenue efficiencies, with the weights being the revenue shares:

$$\overline{RE} = \sum_{k=1}^{K} RE^k \cdot S^k \ , \qquad (3.2.5)$$

where $S^k \equiv \dfrac{py^k}{pY}$ is the revenue share of the unit *k*, and Y is defined as $\sum_{k=1}^{K} y^k$.

The authors also came up with the result that *group technical efficiency* is equal to the weighted sum of individual technical efficiency scores:

$$\overline{TE} \equiv \overline{TE}(x^1,...,x^K,Y) = \sum_{k=1}^{K} TE^k \cdot S^k \ , \qquad (3.2.6)$$

In fact, the relationship (3.2.6) represents *structural technical efficiency* of a group, which is the multi-output extension of the concept intuitively suggested by Farrell (1957) for the single output case.

Analogously to the individual case, *group allocative efficiency* is defined as the ratio of group maximal revenue to the revenue raised from group maximal (*i.e.*, technically efficient) output and equals to the weighted sum of individual allocative efficiencies[7]:

$$\overline{AE}(x^1,..,x^K,Y,p) \equiv \frac{\overline{R}(x^1,..,x^K,p)}{pY \cdot \overline{TE}} \ , \qquad (3.2.7)$$

$$\overline{AE} = \sum_{k=1}^{K} AE^k \cdot S_a^k \ , \qquad (3.2.8)$$

where $S_a^k = \dfrac{py^k \cdot TE^k}{p\sum_{k=1}^{K} y^k \cdot TE^k}$ is the revenue share of *k*-th firm, which is now based on maximal outputs.

[7] In fact, Färe and Zelenyuk (2003) defined it as a ratio of revenue and technical efficiency. The definition (3.2.7) is clearly equivalent, but we believe it is more intuitive.

Finally, using the above definitions, the following decomposition of group revenue efficiency was established:

$$\overline{RE} = \overline{AE} \cdot \overline{TE} . \tag{3.2.9}$$

The results described above proved to be very useful if one needs to infer on the efficiency of the group of decision-making units. Though Farrell (1957) anticipated some of these results almost half a century ago, the issue was given theoretical grounds only recently, by Färe and Zelenyuk (2003). However, some room for improvement still exists, giving an inspiration for further developments. One of such gaps is the issue of efficiency measurement under assumptions of possible *input reallocation across the units*, which we address in the subsequent section.

3.3. MEASURING EFFICIENCY OF A GROUP WHEN INPUTS REALLOCATION IS POSSIBLE

In the preceding section group *output set* is assumed to be equivalent to the sum of individual output sets, according to the definition (3.2.1). Since each unit production set is constructed *given* its individual inputs endowment, it follows that the allocation of inputs among units is fixed under such an aggregation structure. Although the usual assumption is made that given inputs allocation is optimal, one may suspect that if the units operate at non-constant returns to scale, there may be unrealized output gains from inputs reallocation within the group. In addition, if the number of units in the group is relatively constant over time, such underproduction will persist, since decision-making units maximize their own welfare, but not that of the group. In order to allow for such reallocation, we will revise the definition of group technology, closely following the logic of Li and Ng

(1995). We provide their findings in the part (3.3.1) of this section and then develop the extensions, which are presented in the part (3.3.2).

3.3.1. Prior work

Li and Ng (1995) defined group technology, which we denote further as a *group potential technology* in order to stress its difference from the previous definition, as the following:

$$T^g \equiv \{ (X, Y): \ "Y\ can\ be\ produced\ from\ X" \}, \tag{3.3.1}$$

where $Y = \sum_{k=1}^{K} y^k$ and $X = \sum_{k=1}^{K} x^k$.

In words, the definition implies that group technology is defined as a set of all combinations of *aggregate* inputs and outputs, regardless of their distribution across individual units. In contrast to the aggregate technology defined in (3.2.1), this definition allows for reallocation of input-output bundles across the units, and what matters is the aggregate input and respective aggregate output of the group. And this is what we actually need to understand *potential* production possibilities of the group.

To proceed with the central result obtained by the authors, we need to impose additional assumptions on the technologies of individual units:

Assumption 1.

"Individual technologies are identical": $T^k = T, \forall k = 1, \dots, K$

Assumption 2.

"Individual technology sets are convex":

$$(t(x^1,y^1)+(1-t)(x^2,y^2))\in T,\ \ \forall(x^1,y^1),(x^2,y^2)\in T;\ \ t\in[0;1]$$

The authors showed that, when individual technologies are identical (Assumption 1) and convex (Assumption 2), group technology set is equal to K individual technology sets, *i.e.*:

Lemma 2. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2,

$$T^g=\sum_{k=1}^{K}T=KT. \tag{3.3.2}$$

Proof: refer to Li and Ng (1995), pp. 379-380.

Intuitively, what Lemma 2 states is that if (x^k,y^k) is a feasible input-output combination for some unit *k*, then the combination $K\cdot(x^k,y^k)$ is feasible for the group as a whole. Note also, that the Assumption 1 implies that $T^k=T$, $\forall k=1,\dots,K$. So, hereafter we drop the superscript *k* for the notion of individual technology.

The definition of *group potential technical efficiency* is then based on group potential technology T^g:

$$TE^g\equiv TE^g(X,Y)\equiv\max_{\theta}\{\theta:\qquad(X,\theta Y)\in T^g\}. \tag{3.3.3}$$

The authors concluded that group potential technical efficiency is equivalent to the technical efficiency of the "average" unit in the group, *i.e.* an imaginary "representative" unit which employs average amount of group inputs and produces average output[8].

Lemma 3. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2, then

$$TE^{g}(X,Y) = TE(\tilde{x},\tilde{y})^{9}, \tag{3.3.4}$$

where

$$TE(\tilde{x},\tilde{y}) \equiv \max_{\theta}\{\theta : \ (\tilde{x},\theta\tilde{y}) \in T\} \equiv TE\,, \tag{3.3.5}$$

$$\tilde{x} = \frac{1}{K}\sum_{k=1}^{K} x^{k} \ \text{ and } \ \tilde{y} = \frac{1}{K}\sum_{k=1}^{K} y^{k}$$

is the technical efficiency of the average unit.

Proof (also can be found in Li and Ng, 1995):

Using (3.3.2)-(3.3.5), obtain:

$$\begin{aligned}
TE^{g}(X,Y) &\equiv \max_{\theta}\{\theta : (X,\theta Y) \in T^{g}\} = \max_{\theta}\{\theta : (X,\theta Y) \in KT\} = \\
&= \max_{\theta}\{\theta : K^{-1}(X,\theta Y) \in T\} = \max_{\theta}\{\theta : (X / K,\theta Y / K) \in T\} = \\
&= \max_{\theta}\{\theta : (\tilde{x},\theta\tilde{y}) \in T\} \equiv TE(\tilde{x},\tilde{y}).
\end{aligned}$$

Q.E.D.

The authors also define group "*weighted measure of group output technical efficiency*" and "*weighted measure of group output allocative efficiency*", which are equivalent to group structural measures of technical and allocative efficiencies defined by (3.2.6) and (3.2.7).

[8] Noteworthy, this is the idea put forward by Førsund and Hjalmarsson (1979), but was not developed theoretically until the work of Li and Ng (1995). Also note, that group potential technical efficiency defined in (3.3.3) is ≥ 1, which comes from the definition of individual efficiency measure and the fact that group measure is equivalent to efficiency of the average unit. Actually, this also holds for the other the potential group measures established in the subsequent sections of the work.

Li and Ng (1995) then illustrated that group potential output may be larger than the sum of individually technically efficient outputs. Therefore, the measure of the trade-off between these two was proposed, as a residual. So, the authors defined "*output measure of group reallocative efficiency*":

$$REA \equiv \frac{pY \cdot TE^{g}}{\sum_{k=1}^{K} R(x^{k}, p)}, \tag{3.3.6}$$

Here REA is just another notation for RE, originally used by the authors. We use it to distinguish it from revenue efficiency measures. Intuitively, REA indicates how large can be the revenue generated by group potential output, relative to the sum of maximal revenues of individual units.

Finally, the following decomposition was proposed:

$$TE^{g} = \overline{TE} \cdot \overline{AE} \cdot REA, \tag{3.3.7}$$

which comes directly from the definitions of the measures used.

In (3.3.7), we can think of $\overline{AE} \cdot REA$ as of a trade-off between group potential technical efficiency (when reallocation of inputs is possible) and structural technical efficiency of the group, which implicitly assumes that the allocation of inputs within the group is fixed. In fact, REA represents a discrepancy between group potential *technical* efficiency and structural *revenue* efficiency (recall the decomposition (3.2.9)). Such a comparison seems to be not perfectly appropriate, since the underlying units of these measures (output and revenue) are different. Indeed, technical efficiency is concerned with possibility of physical output

---

[9] Hereafter, superscript *"g"* indicates a group measure when reallocation of inputs across the units is possible.

expansion, while revenue efficiency reflects opportunities for the revenue increase.

3.3.2. Our extensions

In this sub-section we develop the extensions which lead to the modifications of Li and Ng (1995) results. In particular, we introduce the notion of group potential revenue efficiency, and decompose it into potential technical and allocative efficiencies. Each of these components are then decomposed into respective measures which allow for inputs reallocation (structural measures), and reallocative efficiency measures, which reflect pure gains from such reallocation. We also show that overall reallocative inefficiency (REA, in Li and Ng notation) may be decomposed into two different components, according to the sources of such inefficiency. The first component is defined as *technical reallocative efficiency*. As we already mentioned in the introductory section, it indicates the *loss in aggregate output* due to non-optimal, in terms of group production, distribution of inputs across units. The second component is *allocative reallocative efficiency*. Despite somewhat confusing wording, this measure makes clear by how much *aggregate revenue* can be increased due to switching to group-optimal *output mix*. It makes sense because group-optimal outputs may imply proportions different to individual ones. In addition, we also show that it is possible to identify each firm contribution to group reallocative (in)efficiency, which was not evident from Li and Ng (1995) findings and may be of particular use in empirical studies. For example, the manager of a multi-unit enterprise can estimate not only efficiency of individual units, but also identify reallocatively inefficient units. Then, it may be the case that inputs may be relatively easily reallocated across these units, so that overall revenues increased. Of course, if such reallocation is costly and involves high transaction, labor, restructuring or other costs, it may be not justified. The decision depends on whether the benefits (higher revenues, or output) overweight such costs. The goal of this section is to develop a measure of these benefits. Note, that we retain the two assumptions made before, *i.e.* convexity and identity of individual technologies.

Potential production possibilities of the group of units

For the purpose of notation consistency, we are going to redefine TE^g with respect to output set, rather then use the definition of Li and Ng (1995), which was based on group potential technology set. So, we begin with the definition of the so called *group potential output set*. The term "potential" is added to stress its difference to the definition of the *group output* set used by Färe and Zelenyuk (2003) (see 3.2.1). An implicit assumption here is that inputs can be reallocated across producers and aggregate output is produced from this "pool" of inputs:

$$P^g \equiv P^g(X) \equiv \{\ y:\ \text{"}all\ y\ producable\ from\ X\text{"}\},\ . \qquad (3.3.8)$$
$$X = \sum\nolimits_{k=1}^{K} x^k,\quad x^k \in \Re_+^N,\quad \forall k = 1,\ldots,K$$

Since $(X,Y) \in T^g$, $P^g(X)$ is *an equivalent characterization of technology* T^g. So, we obtain the following result:

Lemma 4. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2, then it holds that

$$P^g(X) = K \cdot P(\tilde{x}) \qquad (3.3.9)$$

Proof:

Using (3.3.2) and the definition (3.3.8), obtain:

$$P^g(X) = \{y: \ (X,y) \in T^g\} = \{y: \ (X,y) \in KT\} =$$
$$= \{y: \ (X,y)/K \in T\} = K \cdot \{y/K: \ (\tilde{x}, y/K) \in T\} = K \cdot P(\tilde{x})$$

where $P(\tilde{x}) \equiv \{y: \ (\tilde{x}, y) \in T\}$, $\tilde{x} = \frac{1}{K}\sum_{k=1}^{K} x^k$. (3.3.10)

Q.E.D.

That is, group potential output set is equal to K output sets of the average firm, $P(\tilde{x})$, which includes all feasible combinations of outputs which can be produced from the group-average amount of inputs.

Further, we state that $P^g(X) \equiv P^g\left(\sum_{k=1}^{K} x^k\right)$ is less restrictive than $\overline{P}(x^1,...,x^K) \equiv \sum_{k=1}^{K} P(x^k)$:

Lemma 5. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2, the following is true:

$$\overline{P}(x^1,...,x^K) \subseteq P^g\left(\sum_{k=1}^{K} x^k\right). \quad (3.3.11)$$

Proof:

Let $Y^0 = \sum_{k=1}^{K} y^{0,k},\ k = 1,...,K$, such that $Y^0 \in \overline{P}(x^{0,1},...,x^{0,K})$. (*)

Then, $Y^0 \in \overline{P}(x^{0,1},...,x^{0,K}) \Rightarrow Y^0 \in \sum_{k=1}^{K} P(x^{0,k})$,

$\Rightarrow \exists y^{0,k} : y^{0,k} \in P(x^{0,k}), k = 1,...,K, \sum_{k=1}^{K} y^{0,k} = Y^0$

Due to the equivalence of technology characterization by output set and by technology set (Färe and Primont (1995)), we have $y^{0,k} \in P(x^{0,k}) \Leftrightarrow (x^{0,k}, y^{0,k}) \in T,\ \forall k = 1,...,K$.

Hence, $\sum_{k=1}^{K} (x^{0,k}, y^{0,k}) \in \sum_{k=1}^{K} T^k \Rightarrow \left(\sum_{k=1}^{K} x^{0,k}, \sum_{k=1}^{K} y^{0,k}\right) \in \sum_{k=1}^{K} T \Leftrightarrow$ $(X^0, Y^0) \in \sum_{k=1}^{K} T$, and by (3.3.2), we have: $(\mathrm{X}^0, Y^0) \in T^g$. Further, the equivalence of technology characterization implies that $Y^0 \in P^g(X^0)$. (**)

Combining (*) and (**), we finally prove the lemma.

Q.E.D.

The above relationship between potential output set and the sum of individual output sets has an important implications for the group efficiency measurement. It implies that group production possibilities when reallocation of inputs within the group is possible may be wider than if such reallocation is restricted. Hence, even if each unit is efficient and thus operates on the frontier of its individual output set, reallocation of inputs across the units allows for further expansion of the group output, moving it closer to the frontier of potential output set.

Group potential technical efficiency

Due to the equivalence of characterization of technology through the technology set and output set, the result that $TE^g(X,Y) = TE(\tilde{x},\tilde{y})$ still holds if we redefine *group potential technical efficiency* with respect to group potential output set:

$$TE^g \equiv TE^g(X,Y) \equiv \max_{\theta}\{\theta : \quad \theta Y \in P^g(X)\} = \max_{\theta}\{\theta : (X,\theta Y) \in T^g\}. \tag{3.3.12}$$

While our measure of group potential technical efficiency is the same as defined by Li and Ng (1995), we observe that it is different to structural technical efficiency, defined by Färe and Zelenyuk (2003) (see 3.2.6). The difference is due to different assumptions imposed on *group* technology. In case of structural technical efficiency, it is implicitly assumed that inputs allocation across units is fixed, while in the present case it is allowed to vary so that group output is maximized. This leads us to the definition of *group technical reallocative efficiency*.

$$TRE^g \equiv TRE^g(x^1,...,x^K,y^1,...,y^K,K) \equiv \frac{p\left(\sum_{k=1}^{K} y^k\right)\cdot TE^g}{p\sum_{k=1}^{K}\left(y^k \cdot TE^k\right)} \qquad (3.3.13)$$

That is, in words we have:

$$\textit{Group technical reallocative efficiency} =$$

$$= \frac{\textit{technically efficient output of the group (weighted by prices)}}{\textit{output of the group of technically efficient units (weighted by prices)}}$$

This measure gives us the discrepancy between the group potential output and the sum of individually efficient outputs, weighted by corresponding output prices. The nominator is potential group output, *i.e.*, the output value, which can be achieved if inputs are allocated optimally across the units. The denominator reflects group output value when all units are technically efficient and operate independently in sense that inputs can not be reallocated across them. Thus, technical reallocative efficiency represents gains, in terms of group output value, due to reallocation of inputs within this group.

Note that (3.3.13) is different from (3.3.3), the measure defined by Li and Ng (1995), where they have the sum of individual maximal revenues in the denominator, which is somewhat inappropriate, as we have already discussed.

Using the definition (3.2.6) we get the following lemma:

Lemma 6. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2, then

$$TE^g = \overline{TE} \cdot TRE^g \qquad (3.3.14)$$

Proof: follows directly from the definitions (3.2.6) and (3.3.13).

Q.E.D.

That is, TRE^g is a residual between Li and Ng (1995) group *potential* technical efficiency measure and the [*structural*] efficiency measure proposed by Färe and Zelenyuk (2003). Since the measure proposed by the latter work assumes that all units act independently, the intuition provided above is still valid. Stated somewhat differently, TRE^g shows by how much group output can be increased if inputs are reallocated across the units, *given current technical efficiency level of each unit* (which is reflected by structural efficiency, $\overline{TE}$). Note, that in the one output case TRE^g is necessarily ≥ 1, since then $TE^g \geq \overline{TE}$ by Lemma 5. However, multi-output case involves prices as weights in the structural technical efficiency measure, so it may happen that $TE^g < \overline{TE}$. Such an outcome may be somewhat uncomfortable, since it seem to mean that, reallocating the inputs, the group may end up with the lower [efficient] output, comparing to the sum of individually efficient outputs. But this is not so. $\overline{TE}$ involves prices in weights, and thus it does not, in general, measure the distance to the frontier of $\overline{P}(x^1,...,x^K)$ [10]. Therefore it is in principle possible to have $TE^g < \overline{TE}$, while $\overline{P}(x^1,...,x^K) \subseteq P^g\left(\sum_{k=1}^{K} x^k\right)$. But, in practice, we have never observed such a situation. Intuitively, it might happen when some units are just huge, or there products with very large prices, comparing to the other goods[11]. At this stage, we assume that $\overline{TE}$ provides a reasonably good approximation to the distance from the actual output to $\overline{P}(x^1,...,x^K)$, and hence $TE^g \geq \overline{TE}$. Still, as it will be evident further, the inequality $TE^g \geq \overline{TE}$ holds for multi-output case *if* there is no allocative inefficiency among individual units, *or/and* actual output mix of the

[10] For the discussion of this issue and examples the reader may refer to Färe and Zelenyuk (2003).

[11] It should be noted that the question of whether TE^g may be lower than $\overline{TE}$ deserves additional attention. It could be resolved if we reconsider the definition of $\overline{TE}$, as for example, in Bogetoft and Wang (2003). We leave this question for further research.

units is optimal for the maximization of individual revenues *and, simultaneously*, for the maximization of group potential revenue.

Since we are also interested in the *individual unit* analog of the technical reallocative efficiency index, we disaggregate technical reallocative efficiency measure of the group:

> ***Lemma 7.*** If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2,
>
> $$TRE^g = \left(\sum_{k=1}^{K} (TRE^k)^{-1} \cdot S^k\right)^{-1}, \quad (3.3.15)$$
>
> where $TRE^k \equiv \dfrac{TE^g}{TE^k}$. (3.3.16)
>
> *Proof:* using the definition (3.2.6) we have
>
> $$(TRE^g)^{-1} = \frac{p\sum_{k=1}^{K} y^k \cdot TE^k}{p\sum_{k=1}^{K} y^k \cdot TE^g} = \sum_{k=1}^{K} \frac{TE^k}{TE^g} \cdot S^k = \sum_{k=1}^{K} (TRE^k)^{-1} \cdot S^k$$
>
> Taking the inverse from both sides completes the proof.
>
> *Q.E.D.*

Note also that, to a first-order Taylor series approximation around $TRE^k = 1, \forall k = 1,...,K$, (which is a natural point around which our index can be approximated)

$$TRE^g \approx \sum_{k=1}^{K} TRE^k \cdot S^k . \quad (3.3.17)$$

Intuitively, TRE^k is the *technical reallocative inefficiency of an individual unit*, and TRE^g is the weighted average of these individual scores, to a first approximation. The meaning of the definition (3.3.16) is that as far as we can identify technical efficiency of each firm and of the group (which is equal to the technical efficiency of the average firm), TRE^k is also identified, revealing the contribution of each unit to the group technical reallocative efficiency score. Note that if all firms are technically reallocatively efficient, the group also is. TRE^k takes values from zero to infinity, with the optimal value being unity. It may be also seen that even when each unit is technically efficient, TRE^k may be still larger than unity, indicating its technical reallocative inefficiency and resulting in $TRE^g > 1$, which indicate that there are unrealized gains from inputs reallocation. However, the reverse statement can not be done in general. That is, $TRE^g = 1$ does not, in general, imply that each $TRE^k = 1$, because the latter may be greater, equal or less than unity. To see how each TRE^k affects the respective group measure, consider some unit *k*. If it operated at the same scale as the average unit, it could produce $p\tilde{y}^*$ value of output, where $\tilde{y}^*$ is maximum output which can be produced with average inputs. So, if all units produced $p\tilde{y}^*$, technical efficiency of the average unit was 1 and thus the group was technically efficient as a whole, producing $K \cdot p\tilde{y}^*$ with no [technical] reallocative inefficiency. However, if the scales are diverse, $K \cdot p\tilde{y}^*$ may be not achievable without reallocation of inputs, which will be reflected by $TRE^g > 1$. Still, if relatively larger (in terms of revenue shares) units are less technically efficient than smaller units, it is possible to observe $TRE^g = 1$, while some units have TRE^k greater or less than unity.

Group revenue efficiency

In order to define *group potential revenue efficiency*, we should first define *group potential revenue*. It is made in a similar way to (3.2.2). However, potential revenue is defined on group potential output set, $P^g(X)$, rather then on $\overline{P}(x^1,...,x^K)$, as structural revenue was.

$$R^g(X,p) = \max_y \{py : \quad y \in P^g(X)\}. \tag{3.3.18}$$

In words, group potential revenue is the maximal possible group revenue consistent with group potential production possibilities (*i.e.*, group output from the whole "pool" of inputs, thus reallocation is considered possible). It should be clear that $R^g(X,p) \geq \overline{R}(x^1,...,x^K,p)$, which follows directly form the Lemma 5 (3.3.11).

Note, that the Assumption 1 implies that $R^k(x^k,p) = R(x^k,p)$, $\forall k = 1,...,K$. So, we obtain the following equality for the group revenue function:

Lemma 8. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2,

$$R^g(X,p) = K \cdot R(\tilde{x},p) \tag{3.3.19}$$

Proof:

Use the result that $P^g(X) = KP(\tilde{x})$ (see Lemma 4, (3.3.9)) , to show:

$$R^g(X,p) = \max_Y \{pY : \; Y \in P^g(X)\} = \max_Y \{pY : \; Y / K \in P(\tilde{x})\} =$$
$$= K \cdot \max_{Y/K} \{pY / K : \; Y / K \in P(\tilde{x})\} = K \cdot \max_{\tilde{y}} \{p\tilde{y} : \; \tilde{y} \in P(\tilde{x})\} \equiv K \cdot R(\tilde{x},p),$$

where $R(\tilde{x},p)$ is the revenue function of the average firm, by definition.

Q.E.D.

Lemma 8 is quite intuitive, it states that *group potential revenue is equal to K maximal revenues of the average unit.*

Group potential revenue efficiency is then defined as the ratio of group potential revenue to actual group revenue:

$$RE^g(X,Y,p) \equiv \frac{R^g(X,p)}{pY}. \tag{3.3.20}$$

From the Lemma 8 and (3.3.20), it follows that *group revenue efficiency is equivalent to the revenue efficiency of its average unit*, as we state in the following lemma.

Lemma 9. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2,

$$RE^g(X,Y,p) = RE(\tilde{x},\tilde{y},p) \tag{3.3.21}$$

Proof:

We use Lemma 8 (3.3.19) to show this.

$$RE^g(X,Y,p) \equiv \frac{R^g(X,p)}{pY} = \frac{K \cdot R(\tilde{x},p)}{pY} = \frac{R(\tilde{x},p)}{pY\,/\,K} = \frac{R(\tilde{x},p)}{p\tilde{y}} \equiv RE(\tilde{x},\tilde{y},p)\,.$$

Q.E.D.

Similarly to group technical reallocative efficiency in (3.3.13), we define *group revenue reallocative efficiency* as the discrepancy between group potential revenue and the sum of individual maximal revenues[12]:

$$RRE^g \equiv \frac{\left(\sum_{k=1}^{K} py^k\right) \cdot RE^g}{\sum_{k=1}^{K}\left(py^k \cdot RE^k\right)}, \tag{3.3.22}$$

Or, expressed in words:

$$\textit{Group revenue reallocative efficiency} = \frac{\textit{group potential revenue}}{\textit{revenue of the group of revenue efficient units}}$$

The following theorem immediately follows from the definition of group revenue reallocative efficiency (3.3.22):

---

[12] Throughout the text we sometimes name this measure as s "group reallocative efficiency" (without the word "revenue"), or "overall reallocative efficiency", since it represents *overall* reallocative efficiency gains to the group.

Theorem 1[13]. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2,

$$RE^g = \overline{RE} \cdot RRE^g , \tag{3.3.23}$$

Proof: the result follows from (3.2.5) and (3.3.22).

Q.E.D.

Theorem 1 indicate that RRE^g is the residual between group *potential revenue efficiency* and group *structural revenue efficiency*. In other words, it shows the discrepancy between group efficiency when inputs can be reallocated and group efficiency when such reallocation is not possible. In other words, it shows by how much group revenues can be increased if inputs are reallocated across the units, *given current revenue efficiency level of each unit.*

Further, using the decomposition (3.2.9), established by Färe and Zelenyuk (2003), we obtain the decomposition of group potential revenue efficiency, which we state in the following corollary to the corollary:

Corollary 1. Theorem 1 implies that the following decomposition holds:

$$RE^g = \overline{TE} \cdot \overline{AE} \cdot RRE^g . \tag{3.3.24}$$

Proof: the proof is done by substituting in (3.3.23) $\overline{RE}$ with the right-hand side of the decomposition (3.2.9).

Q.E.D.

The decomposition in (3.3.24) may contrasted to the one obtained by Li and Ng (1995) (see 3.3.7). The difference is that now we have revenue efficiency measure

[13] In fact, the same relationship was established by Coelli et al. (2003). The shortcomings of their work have been discussed in the Literature Review section.

on the left-hand side, making the link between two sides more logical. That is, overall (revenue) efficiency is decomposed into respective structural measures and reallocative component, which identifies gains from reallocation.

Note that Theorem 1, together with Lemma 5 (3.3.11), implies that $RRE^g \geq 1$:

> ***Corollary 2.*** $RRE^g \geq 1$, with the strict equality being achieved if and only if $RE^g = \overline{RE}$.
>
> *Proof:*
> Using Lemma 5 (3.3.11) and the definitions (3.2.4) and (3.3.18), obtain:
> $\overline{P}(x^1,...,x^K) \subseteq P^g\left(\sum_{k=1}^{K} x^k\right)$ (Lemma 5) $\Rightarrow R^g(X,p) \geq \overline{R}(x^1,...,x^K,p)$,
> which in turn implies that $(RRE^g = RE^g / \overline{RE}) \geq 1$. Finally, using Theorem 1 (3.3.23), $RRE^g = 1$ if and only if $RE^g = \overline{RE}$.
> *Q.E.D.*

As was mentioned above, individual revenue maximization does not imply maximization of the group revenue, if inputs can be reallocated within the group. In such a case it is likely that $RRE^g > 1$. However, if individual optimizing goals are *in accord* to group ones, RRE^g will be equal to 1, which seem to be rare in practice, though theoretically possible.

In addition, we disaggregate group revenue reallocative efficiency, in order to get relative individual unit's contribution to the group score. We do it similarly to the disaggregation of technical reallocative efficiency of the group (see Lemma 7 (3.3.15)):

> ***Lemma 10.*** If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2,

$$RRE^{g} = \left(\sum\nolimits_{k=1}^{K} (RRE^{k})^{-1} \cdot S^{k}\right)^{-1}, \tag{3.3.25}$$

$$\text{where } RRE^{k} \equiv \frac{RE^{g}}{RE^{k}}. \tag{3.3.26}$$

Proof: using the definition (3.3.22), we have

$$(RRE^{g})^{-1} = \frac{\sum\nolimits_{k=1}^{K} py^{k} \cdot RE^{k}}{\sum\nolimits_{k=1}^{K} py^{k} \cdot RE^{g}} = \sum\nolimits_{k=1}^{K} \frac{RE^{k}}{RE^{g}} \cdot S^{k} = \sum\nolimits_{k=1}^{K} (RRE^{k})^{-1} \cdot S^{k}$$

Taking the inverse from both sides completes the proof.

Q.E.D.

Similar to what we have shown above for the case of group technical reallocative efficiency, $\sum\nolimits_{k=1}^{K} RRE^{k} \cdot S^{k}$ is a first-order approximation to RRE^{g} around $RRE^{k} = 1, \forall k = 1,...,K$:

$$RRE^{g} \approx \sum\nolimits_{k=1}^{K} RRE^{k} \cdot S^{k}. \tag{3.2.27}$$

Intuitively, RRE^{k} here represents *technical reallocative inefficiency of an individual unit*, and, to a first approximation, RRE^{g} is the weighted average of these individual scores. RRE^{k}, defined in (3.2.26), reveals the contribution of each unit to the group revenue reallocative efficiency score. Note *that if all firms are revenue reallocatively efficient, the group also is.* RRE^{k} takes values from zero to infinity, with the optimal value being unity. Again, even when each unit is revenue efficient, RRE^{k} may be still larger than unity, indicating its revenue reallocative inefficiency and resulting in $RRE^{g} > 1$. Similar to the case of technical reallocative efficiency, the reverse statement can not be done in general. So, $RRE^{g} = 1$ does not, in general, imply $RRE^{k} = 1, \ \forall k = 1,...,K$. Again, it may happen if larger units (in terms of revenue shares) are relatively less revenue efficient than smaller units.

The idea of group revenue reallocative efficiency may be also illustrated graphically.

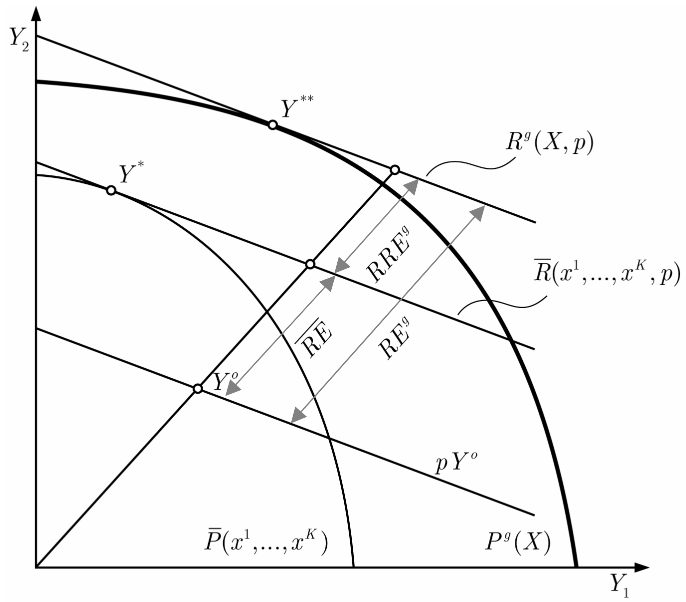

Figure 1. Group revenue reallocative efficiency

On the Figure 1 above, the line $\overline{R}(x^1,...,x^K,p)$ corresponds to group maximal revenue function defined on group output set and is equal to the sum of individual revenue functions (3.2.3). It presumes impossibility of inputs reallocation. In contrast, $R^g(X,p)$ indicates group *potential* revenue, *i.e.* maximum group revenue when inputs can be reallocated. Clearly, even if $\overline{R}$ is achieved at the output bundle Y^*, it may be increased further to R^g which corresponds to group output Y^{**}. The trade-off between $\overline{R}$ and R^g is attributed to non-optimal inputs allocation within the group, in terms of potential revenue maximization. This trade-off will be eliminated if inputs are allocated in an optimal way in terms of group output, so the group becomes revenue reallocatively efficient and $RRE^g = 1$.

Group allocative efficiency

The idea of group allocative efficiency has similar intuition to the analogous individual measure in (3.1.6) and the "structural" measure defined in (3.2.7). However, now we have to compare group potential [maximal] revenue (when reallocation of inputs is possible) to that obtainable from potential group output (also, when reallocation is possible). We define this measure as *group potential allocative efficiency*:

$$AE^g(X,Y,p) \equiv \frac{R^g(X,p)}{pY \cdot TE^g} \tag{3.3.28}$$

Similarly to the way it has been done for technical and revenue efficiencies, we show that *group potential allocative efficiency is equivalent to the allocative efficiency of the average unit*:

Lemma 11. If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2,

$$AE^g(X,Y,p) = AE(\tilde{x},\tilde{y},p). \tag{3.3.29}$$

Proof: Using Lemma 3 (3.3.4) and Lemma 8 (3.3.19), obtain:

$$AE^g(X,Y,p) \equiv \frac{R^g(X,p)}{pY \cdot TE^g} = \frac{K \cdot R(\tilde{x},p)}{pK \cdot \tilde{y} \cdot TE} = \frac{R(\tilde{x},p)}{p\tilde{y} \cdot TE} \equiv AE(\tilde{x},\tilde{y},p),$$

where the last equivalence follows from the definition of individual allocative efficiency in (3.1.6).

Q.E.D.

Group allocative reallocative efficiency is then defined as a residual between AE^g and respective "structural" measure, $\overline{AE}$:

$$ARE^g \equiv \frac{AE^g}{\overline{AE}}. \qquad (3.3.30)$$

That is, ARE^g represents relative difference between group potential allocative efficiency (with inputs reallocation possible) and structural allocative efficiency measure proposed by Färe and Zelenyuk (2003) (which does not allow for inputs reallocation). Note that $0 < ARE^g < \infty$, since AE^g may be larger, smaller or equal to $\overline{AE}$. These two latter measures close the relationships between structural and potential revenue and technical efficiencies, respectively, and hence are based on the different assumptions about possibility of inputs reallocation. Thus, they should not be expected to be related in the form of one-sided inequality. Accordingly, ARE^g may be greater, equal, or less than 1. If $ARE^g > 1$, it means that allocation of resources to each output Y_m, $m = 1,...,M$, is *less optimal* in terms of *group* revenue maximization (so, when inputs reallocation is possible) than it is in terms of maximization of the sum of individual revenues (*i.e.*, when inputs reallocation is not possible). If inputs can be reallocated *across units*, and if they are appropriately allocated across activities *within the group*, so that *group* revenue is being maximized, AE^g goes to 1. But not necessarily $\overline{AE}$ does, since it reflects optimality of the output mix (and thus input mix) for the *individual unit*, which may be inconsistent with group-wise optimization. Thus, we may end up with the case when $ARE^g < 1$. Accordingly, if it happens that group-optimal allocation of individual resources to each activity is *also optimal* from the point of view of individual unit (or *vice versa*), $ARE^g = 1$. Perhaps more intuitively, this measure can be thought of as indicating the extent to which individual units have to adjust their input mixes in case of reallocation of inputs across the units. Note that *current input mixes are taken by the measure as given* and thus may be individually not optimal (not maximizing individual revenue). We will provide more intuition for this measure further.

It follows that the decomposition (3.2.9), which was stated for weighted measures, is preserved for group potential efficiency measures.

***Proposition**:* If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2

$$RE^g = TE^g \cdot AE^g \tag{3.3.31}$$

Proof:

The proof is done by recalling (3.1.7) and the fact that all the components in the decomposition are equal to the respective individual measures for the average unit.

Q.E.D.

Hence, if one wants to preserve the possibility of inputs reallocation within the group and thus turns to the *potential* measures of group efficiency, he or she still is able to decompose group efficiency, now potential one, into technical and allocative components. The particular advantage of these measures and the decomposition above is that the information they provide is wider than one can obtain from the structural measures. If reallocation of inputs across the units is possible, it consistently reflects the efficiency of the group. If such reallocation is not possible, each component of this measure may be decomposed and respective structural measure can be thus extracted. The other advantage is that, in the above decomposition, its technical part does not depend on price information and hence represents *pure* technical efficiency of the group. Moreover, since the measures of potential efficiency of the group (*i.e.*, when reallocation of inputs is possible), are proved to be equivalent to the respective individual measures for the average unit*, these group measures inherit all desirable* *properties, which Farrell-type measures satisfy*. In this respect, these measures are superior to the other group measures. Yet, to use these measures one should verify underling assumptions, which may be restrictive in some cases.

Now we turn back to the measure of group allocative reallocative efficiency, and decompose it into respective individual measures. We do it in a similar way we did it before for the respective technical and revenue efficiencies (see Lemma 7 and 10):

> ***Lemma 12.*** If technology satisfies Regularity Axioms 1-5 and assumptions 1 and 2,
>
> $$ARE^g = \left(\sum_{k=1}^{K} (ARE^k)^{-1} \cdot S_a^k\right)^{-1}, \qquad (3.3.32)$$
>
> where $ARE^k \equiv \dfrac{AE^g}{AE^k}$. (3.3.33)
>
> *Proof:* using the definitions (3.2.7) and (3.3.32), we have
>
> $$(ARE^g)^{-1} = \frac{\sum_{k=1}^{K} (py^k \cdot TE^k) \cdot AE^k}{\sum_{k=1}^{K} (py^k \cdot TE^k) \cdot AE^g} = \sum_{k=1}^{K} \frac{AE^k}{AE^g} \cdot S_a^k = \sum_{k=1}^{K} (ARE^k)^{-1} \cdot S_a^k$$
>
> Taking the inverse from both sides completes the proof.
>
> *Q.E.D.*

Similar to what we have shown above for the case of group technical and revenue reallocative efficiency, $\sum_{k=1}^{K} ARE^k \cdot S_a^k$ is a first-order approximation of ARE^g around $ARE^k = 1,\ \forall k = 1,\ldots,K$:

$$ARE^g \approx \sum_{k=1}^{K} ARE^k \cdot S_a^k. \qquad (3.3.34)$$

In words, ARE^k can be thought of as a measure of *individual allocative reallocative efficiency* and represents the loss/gain of unit's revenue due to utilizing non-optimal (not maximizing group revenue) combination of inputs. While ARE^g is a weighted average of the individual measures, to a first approximation. As in the previous sections, the decomposition (3.3.32) gives us the result that $ARE^g = 1$ if each unit is allocatively reallocatively efficient. That is, if allocation $(x_1^k, ..., x_n^k)$ is optimal for each unit *k*, *and simultaneously* the allocation of total inputs $(X_1, ..., X_n)$ is optimal for the group revenue maximization, each ARE^k is equal to 1, and there is no allocative reallocative inefficiency on the group level either. Again, the reverse is not necessarily true. The input mixes (and thus output mixes) of some units may be consistent with group revenue maximization, but the allocations of others may not. Respectively, ARE^k may be equal, less or greater than unity, and it may result in $0 < ARE^g \leq 1$ or $ARE^g > 1$. If, in aggregate, larger units are more allocatively inefficient than the smaller ones, ARE^g is likely to exceed unity. In the opposite situation, it may be less than unity. Though, if $ARE^g = 1$ and we observe that some units have $ARE^k > 1$, and some have $ARE^k < 1$, it simply means that, if the possibility of inputs reallocation exists, aggregate allocative inefficiency of the former units is actually higher (then if reallocation was not possible), while the aggregate allocative inefficiency of the latter is in fact lower. Roughly speaking, you can think of this as of a situation when the subgroup of units with $ARE^k > 1$ have $AE^{g,1} > \overline{AE}^1$ and for the other subgroup, where all units have $ARE^k < 1$, the reverse holds, *i.e.* $AE^{g,2} < \overline{AE}^2$ (superscripts 1 and 2 denote subgroups). Though we did not define subgroup measures, such an explanation may be helpful for understanding the intuition behind the concept with a confusing name "allocative reallocative efficiency". To summarize the intuition above, which was expressed in different ways, we state that, ARE^g *reflects an effective mismatch between individual and group input (or, equivalently, output) mixes*. We say *effective*, because such a mismatch may not necessarily result in $ARE^g \neq 1$, that is, it does not necessarily lead to allocative reallocative inefficiency at the group level.

Decomposition of group reallocative efficiency

From the discussion in the above sections, one may have noticed that reallocative inefficiency may come from different sources. It may indicate revenue gains due to reallocation of inputs (RRE^g), output gains (TRE^g), or represent an effective mismatch between group and individual input mixes (ARE^g). Are these measures somehow connected to one another? It turns out that the relationship exist and is very close and intuitive. We state it as a theorem.

Theorem 2. If technology satisfies Regularity Axioms 1-5, assumptions 1 and 2, and if inputs can be reallocated within the group of units, then there may exist overall group efficiency gains due to inputs reallocation, which can be decomposed as:

$$RRE^g = TRE^g \cdot ARE^g, \tag{3.3.35}$$

Proof:

Applying the result (3.2.9), Lemma 6 (3.3.14), the definition (3.3.22), the Theorem 1 (3.3.23) to the Proposition (3.3.31) gives the desired result.

Q.E.D.

So far, we have reached our goal of the decomposition of overall (revenue) reallocative efficiency of the group of units into two components: technical reallocative efficiency and allocative reallocative efficiency. The relative magnitude of these components reveals whether the goals of the firm (unit) are more consistent with the maximization of group output or group revenue, or both. If each unit produces group revenue maximizing output mix and also utilizes group output maximizing input mix, then the *allocation* of economic resources *within* the group is perfect and $RRE^g = 1$. RRE^g is ≥ 1, so there are possible combinations of TRE^g and ARE^g, which lead to RRE^g being strictly greater than 1, as well as it is possible to have $RRE^g = 1$, while the other two components are not.

To conclude this section, we consider the intuition behind some cases, which involve not always easy interpretation at the first sight.

1) $RRE^g > 1,\ TRE^g > 1\ and\ ARE^g < 1$: (a) Even if each unit is individually revenue efficient, overall group revenue can be increased due to inputs reallocation across the units. (b) Units are more successful in achieving higher individual technical efficiency relative to the achieving group potential technical efficiency. Reallocation of inputs across units can increase total group output. (c) Current allocation of the resources within the units (*i.e.*, individual input-mixes) is more beneficial for maximizing group potential revenue (reallocation is possible) rather than for maximizing [the sum of] individual revenues (reallocation is not possible). So, there is an alternative input mix which can contribute to achieving group potential revenue. However, in this case it is relatively easier to change input-mix, if maximum group potential revenue is the goal, rather than if the goal is to maximize the sum of individual revenues.

2) $RRE^g > 1,\ TRE^g > 1\ and\ ARE^g > 1$: (a) and (b) the same. (c) Current allocation of the resources within the units (*i.e.*, individual input-mixes) is more beneficial for maximizing the sum of individual revenues rather than for maximizing group potential revenue. It is relatively easier to change input-mix, if maximum sum of individual revenues is the goal, rather than the goal is to maximize group potential revenue.

3) $RRE^g = 1,\ TRE^g > 1\ \ and\ ARE^g < 1$: Even if each unit is perfectly efficient, if reallocation of inputs across the units is allowed, group output can be increased further. However, such an increase due to reallocation can not be done without altering individual input mixes of some units, which offsets revenue gains from the increase in group output.

4) If $RRE^g > 1$ and $TRE^g\ \ or\ \ ARE^g$ equals to 1, then the potential revenue inefficiency comes solely from the one of these two components.

The other cases can be interpreted using similar logic to the used above.

Chapter 4

SIMULATED EXAMPLE

In this chapter, we provide the exposition of how reallocation of inputs across the units may enhance overall efficiency of an enterprise using simulated production process. Although it would be interesting to test our theoretical conclusions on real life data, there are some reasons to use the simulation. First, an estimation always involves some kind of distortions and estimates are *per se* an approximation to true parameters. In order to avoid these distortions in our exposition, we use complete information about the population, as well as true data in a sense that the data generating process is perfectly known. Second, the data on the performance of units within the enterprise is a management accounting data, which is often hard to get. As in our case, when we approached the Kyiv office of the Aval Bank in search of such data. Therefore, in this work we employ a simulation method. Our hypothetical enterprise consists of ten units producing two outputs from one input. In what follows, we begin with the description of the data generating process and proceed with the demonstration of efficiency gains from inputs reallocation.

4.1. DATA GENERATING PROCESS AND INDIVIDUAL EFFICIENCIES

As it was already mentioned, our hypothetical enterprise and its units produce two outputs using one input. The enterprise is assumed to have a horizontal structure. That is, different units represent production centers which do not use the outputs of the other units as inputs. For example, one may imagine that there is a warehouse, which provides all the units with [homogeneous] inputs. It is also assumed, as it has been done in the theoretical part of the work, that all units have the same technology. The stochasticity in our example comes form the random nature of the scales of operation and the randomness of the output mixes. Below we proceed with the detailed description of the data generating process[14].

Inputs

In order to randomly alter the outputs proportion (y_1^{*k} vs y_2^{*k}) for each unit, we let some portion of singe input x^k to be used for the production of y_1^{*k} and the other portion – for the production of y_2^{*k}. Note that in the * in the subscripts signifies that, as yet, we consider efficient output levels (inefficiency will be introduced further). So, the inputs are generated for each unit separately, according to the following rule.

$\alpha x^k = x^{k,1}$ (input share for the production of y_1^{*k})

$(1-\alpha)x^k = x^{k,2} = x^{k,1} \cdot (0.7 + e^k)$ (input share for the production of y_2^{*k})

$e^k \sim N^+(0,1)$, $\alpha \in [0;1]$, $k = 1,\ldots,10$.

That is, first we generate input portion to be used to produce y_1^{*k}, and then the other portion of it is used to produce y_2^{*k}. Note, that we introduce some degree of complementarity of outputs, in sense that extent of the production of y_2^{*k} depends on the level of the first output, plus stochastic error. This is quite

[14] The list of all MatLab codes and data files is provided in the Appendix 1. The files are available on EERC server in the data folder \\Kitty\MATheses\MATheses2004\vnesterenko.

common in real life. The reader should also note that the input usage and hence the scale of activity is determined primarily by the amount of the input used for the y_1^{*k} production, $x^{k,1}$. It is generated in a way, which allows us to make some units large, and some of them small. It would be helpful for our exposition, since the materiality of efficiency gains depends also on how different are the units' sizes. So, $x^{k,1}$ is generated as the following:

$$x^{k,1} = \begin{cases} 10 + 5 \cdot 10^4 \cdot \varepsilon^k, & key^k \leq 0.5 \\ 10 + 100 \cdot 10^4 \cdot \varepsilon^k, & key^k > 0.5 \end{cases},$$

$$k = 1,...,10 .$$

Above, ε^k is a random term distributed uniformly from 0 to 1, which makes $x^{k,1}$ also uniformly distributed over the interval $[10; 5 \cdot 10^4]$. The variable key^k is a random number, from 0 to 1, which is used to randomly switch between generation of small and large units. We also presume some fixed cost, equal 10, which the unit can not avoid if it starts operating. We present here the summary for the generated inputs, and the detailed data can be found in the Appendix 2.

Table 1. Summary of generated inputs

	Proportion of x^k used to produce y_1^{*k} $(x^{k,1} / x^k)$	Proportion of x^k used to produce y_2^{*k} $(x^{k,2} / x^k)$	Share of the unit in overall input usage
Total (over all units)	0.4610	0.5390	1.0000
Average	0.4697	0.5303	0.1000
Max	0.5460	0.6998	0.3341
Min	0.3002	0.4540	0.0003

So, as it can be inferred from the table above, on average the units use relatively more of input x to produce y_2^{*k}, and their scales are very disperse, as it was desired.

Outputs

Now we proceed with the description of production process, that is, how these inputs are transformed into respective outputs. We assume a simple Cobb-Douglas technology, which is the same for all units. It is also assumed to be the same for each of the outputs, for simplicity. So, the production function is:

$$y^{*k} = f(x^k) = (y_1^{*k}, y_2^{*k})',$$

$$\text{where } y_m^{*k} = f(x^{k,m}) = 500 \cdot (x^{k,m})^{0.5}, \qquad m = 1,2;\ k = 1,\dots,10 .$$

Note that $x^{k,m}$ denotes the portion of the *single* input x^k, which is used to produce either y_1^{*k} or y_2^{*k}. Note also, that these outputs are efficient in Farrell sense, because they are on the production frontier represented by $f(x^k)$. The technology is convex and identical for all units, so assumptions 1 and 2 are satisfied. Clearly, regularity conditions are also satisfied.

Prices and revenues

Consistent with the assumption made in the theoretical part, prices are assumed to be the same for all units. For the sake of simplicity, we let them to be also the same for both outputs (without loss of generality): $p = (p_1, p_2) = (1,1)$. Obviously, $p \cdot y^{*k}$ may be not the maximal revenue of the unit. So as to obtain the value of maximal revenue function, given inputs and prices, we solve the following revenue maximization problem for each unit:

$$R(x^k, p) = \max_{x^k}\{pf(x^k):\ x^{k,1} \le (x^{k,1} + x^{k,2})\}, \qquad k = 1,\dots,10.$$

The problem is solved by `fmincon` MatLab routine. As a result, we obtain maximum individual revenues.

"Observed" outputs, revenues and individual efficiency

Since we want to analyze *efficiency* of the enterprise and its constituent units, it is necessary to make some units perform inefficiently. So, we randomly generated the output technical efficiency scores for the units:

$$TE^k = 1 + \xi^k, \quad \xi^k \sim N^+(0,1), \quad k = 1,\dots,10.$$

In turn, $y^k = y^{*k} / TE^k$ gives the vector of "observed" output levels for each unit. Respectively, $p \cdot y^k$ gives the "observed" individual revenues. Once we have them, we can compute individual revenue efficiencies, as $RE^k = R(x^k, p) / py^k$ (see 3.1.4). According to (3.1.6), the ratio $RE^k / TE^k = AE^k$ represents individual allocative efficiency. In the table below, we provide the data on individual outputs, revenues and efficiency scores.

Table 2. Individual outputs, revenues and efficiency scores*

k	y_1^k	y_2^k	y_1^{*k}	y_2^{*k}	$p \cdot y^k$	$R(x^k, p)$	RE^k	TE^k	AE^k
1	276190	281810	495070	505150	558000	1000300	1.7926	1.7925	1.0001
2	37494	57241	65640	100210	94736	169420	1.7883	1.7507	1.0215
3	51695	57901	62891	70441	109600	133550	1.2185	1.2166	1.0016
4	99868	94991	132890	126400	194860	259380	1.3311	1.3307	1.0003
5	14656	13712	15834	14815	28368	30666	1.0810	1.0804	1.0006
6	130160	131880	158220	160310	262040	318540	1.2156	1.2156	1.0000
7	111510	109860	183040	180320	221360	363370	1.6415	1.6414	1.0000
8	21279	20278	45326	43194	41557	88545	2.1307	2.1301	1.0003
9	161650	228220	374210	528310	389870	915590	2.3484	2.3149	1.0145
10	353170	322030	467320	426120	675190	894400	1.3246	1.3232	1.0011

* hereafter, the computations are done using the MatLab® 6.5 software. The codes are available on the EERC server data folder \\Kitty\MATheses\MATheses2004\vnesterenko.

The data suggests that there are a lot of inefficient units, and the inefficiency is attributed mostly to the technical inefficiency. Thus, output mixes are relatively more consistent with revenue maximization, but the levels of output can be increased further, by about 8.1% for the most technically efficient unit, to approximately 134% for the least efficient. Now we turn to the estimation of the efficiency of the enterprise as a group of units.

4.2. GROUP EFFICIENCY MEASUREMENT

We begin with computing group structural efficiency measures, which presume that inputs can not be reallocated across the units. Then, we will compute respective potential and reallocative efficiency measures in order to see whether group output/revenue can be increased if inputs are reallocated.

Structural efficiency

Using the definitions of structural efficiencies from the section 3.2, we obtain the following results:

Table 3. Group structural efficiency*

Structural revenue efficiency, $\overline{RE}$	1.6205
Structural technical efficiency, $\overline{TE}$	1.6135
Structural allocative efficiency, $\overline{AE}$	1.0043

* revenue weights were assumed

These scores are perfectly consistent with the individual measures computed before. As individual scores do, they suggest that the inefficiency of the group should be attributed generally to the individual technical inefficiencies, rather than to inappropriate output mixes (allocative component, $\overline{AE}$, is relatively low). In addition, the measure of group overall (revenue) efficiency may be compared with the simple average of individual scores, which is 1.5872. Since the structural measure is the *weighted* average of individual score, with revenue shares as weights, one may conclude that there are relatively large inefficient units in the group, which is really so (e.g., units 1, 7 and 9). The results suggest that to make the group efficient, the manager of the enterprise should aim at increasing the technical efficiency of its constituent units. If they all were perfectly technically and allocatively efficient, the enterprise would have been efficient as well. But this is necessarily so only in case inputs can not be reallocated within the group.

Group potential and reallocative efficiency measures

Now we want to see what is the efficiency of the enterprise, if inputs can be reallocated across the units. It is also interesting whether group output/revenue may be increased further, even if individual units are perfectly efficient. The first question is addressed by calculating potential efficiency measures of the group. The second one – by calculating reallocative efficiency. We do this as it was proposed in the section 3.3. The results are provided in the Table 4, where structural measures are also included for the expository purpose.

Table 4. Structural, potential and reallocative efficiency of the group

	Potential	Structural	Reallocative
Revenue efficiency	RE^g	$\overline{RE}$	RRE^g
	2.1248	1.6205	1.3112
Technical efficiency	TE^g	$\overline{TE}$	TRE^g
	2.1242	1.6135	1.3165
Allocative efficiency	AE^g	$\overline{AE}$	ARE^g
	1.0003	1.0043	0.9960

It can be seen from the Table 4 that, (with no inputs reallocation) if all units performed technically efficiently, the output could be increased by about $\overline{TE} = 1.6135$ (≈61%) in money terms, and if they become revenue efficient, revenues can be increased by about $\overline{RE} = 1.6205$ (≈112%). However, we observe that the group potential efficiency measures indicate lower efficiency than structural measures do. Specifically, $RE^g = 2.1248 > \overline{RE} = 1.6205$. Hence, one may look at reallocative efficiencies to see that if inputs are reallocated across *all* units, overall group output can be increased by additional $TRE^g = 1.3165$ (≈32%) and revenues can be increased by $RRE^g = 1.3112$ (≈31%). The measure of potential allocative efficiency, AE^g, is larger than the respective structural measure, $\overline{AE}$, which results in $ARE^g = 0.9960 < 1$. This is actually interesting case. Both allocative efficiency measures have values close to unity, which means that the individual output mixes are almost optimal, either in sense of individual revenue maximization ($\overline{AE}$ is close to 1), or in sense of maximizing group potential revenue (AE^g is close to 1). However, insignificantly lower than 1 value of ARE^g indicates that, on aggregate, current individual output mixes are a bit more consistent with group potential revenue ($R^g(X,p)$) maximization, rather than with maximization of the sum of individual revenues ($\sum_{k=1}^{K} R(x^k, p) = \overline{R}(x^1,...,x^K, p)$). So, if reallocation of inputs is possible, there is almost nothing to gain (in terms of group revenue maximization) from altering output mixes of individual units. Respective increase in the absolute values of group output and revenue due to enhanced efficiency (including reallocative efficiency) can be inferred from the Table 5.

Table 5. Implied total output/revenue gains from reallocation of inputs*

Actual total output in money terms	\$ 2 585 595
WITH NO INPUTS REALLOCATION:	
Maximum total output in money terms	\$ 4 155 711

(*i.e.*, if all units are technically efficient)	
The sum of maximum revenues (if all units are revenue efficient)	\$ 4 173 761
IF INPUTS ARE REALLOCATED:	
Potential output in money terms, $pY \cdot TE^g$ (if potential technical efficiency is 1, that is, if all units are technically efficient and output gains from reallocation are realized)	\$ 5 471 079
Potential maximal revenue, $pY \cdot RE^g$ (if potential revenue efficiency is 1, that is, all units are revenue efficient and revenue gains from reallocation are realized)	\$ 5 472 624
Possible increase in the total revenue, solely from the inputs reallocation across the units $4\,173\,761 \cdot (1 - RRE^g)$	+ \$ 1 298 863

* the above results may not perfectly coincide with the reader's ones, due to the rounding error

So, pure potential revenue gains due to the reallocation of inputs across the units may be quite substantial. In this example they amount to approximately 31%, or about \$1.3 million of extra revenues potentially available if inputs are reallocated.

However, at this stage it is not practically visible what would happen to these measures when we (or the manager of an enterprise) actually reallocate inputs across the units. Moreover, it would be interesting to see what happens if the inputs are reallocated across not all, but only some of the units, since the process is very likely to be sequential in practice. In addition, reallocation across some units may lead to relatively greater enhancement of group efficiency, than across the others. We will address these issues in the section that follows.

4.3. Reallocating the inputs

So far, we are armed with the theoretical results and know how to infer on the group efficiency. However, to understand the practical applications, we are going to actually reallocate inputs and see what happens.

First, we perform the reallocation across *all* the units of the group. Note that the individual efficiencies are held constant. The results are presented in the Table 6.

Table 6. Group efficiency after the reallocation of inputs

	Potential	Structural	Reallocative
Revenue efficiency	RE^g	$\overline{RE}$	RRE^g
	1.4899	1.4899	1.0000
Technical efficiency	TE^g	$\overline{TE}$	TRE^g
	1.4887	1.4887	1.0000
Allocative efficiency	AE^g	$\overline{AE}$	ARE^g
	1.0008	1.0008	1.0000

As the theory suggests, all measures of reallocative efficiency indicate that the gains from reallocation are seized (they are equal to 1). The remaining portion of inefficiency is attributed entirely to the respective structural components. So now our hypothetical manager should address the problem of improving individual efficiencies, so as to achieve potential output/revenue level.

However, it may be not very realistic to reallocate the inputs across all the units at once. More likely, the process would involve 2 or, say, 3 units at the same time. Furthermore, as we have already mentioned, the effect of partial reallocations is not obvious *apriori* when relatively larger units are also relatively more efficient. This issue deserves closer attention. Let us consider 3 possible cases.

Case 1. All units are individually revenue efficient. In this case $\overline{RE} = 1$. Since $RRE^g = RE^g / \overline{RE}$, we will have $RRE^g = RE^g$. Moreover, since $RE^g = R^g(X, p) / pY$, reallocation from larger to smaller units would lead to the increase in pY, due to decreasing returns of larger units and increasing returns of the smaller ones (we have variable returns to scale technology). In effect, RRE^g and RE^g would decrease, meaning that the efficiency improves. In other words, *efficiency effect* on RE^g is absent and there is only *scale effect*.

Case 2. Larger units are relatively *less* efficient than the smaller ones. In such a case reallocation from large to small units would lead to the decrease in $\overline{RE}$ (obvious) and also in RRE^g (as explained above), and thus in RE^g. So, *efficiency effect* would be working in the same direction as *scale effect*, improving overall group efficiency.

Case 3. Larger units are relatively *more* efficient than the smaller ones. There is a problem. When we reallocate, $\overline{RE}$ increases, but RRE^g decreases. So, the outcome for RE^g depends on both, relative sizes and individual efficiencies, meaning that *efficiency effect* works in the opposite direction to *scale effect*. Hence the possibility exist, and we will show that this is what actually happens, that partial reallocation (i.e., across some of the units) may be not beneficial for the group.

Therefore, in order to understand possible effects of partial adjustments better, we simulate pair-wise reallocations. After each iteration we recalculate the efficiency scores, the individual ones and for the group as a whole. The logic suggests that the reallocation which leads to the improvement of group potential efficiency should be considered as beneficial, and harmful otherwise. Since we have only ten units, there are 45 possible combinations of pair-wise reallocations. For the sake of brevity, we provide the results only for 10 of those reallocations, 5 of which led to the largest improvements of potential revenue efficiency of the group, and the other 5 which resulted in the least improvements or even worsening the group efficiency. The results are presented in the table below.

Table 7. The most and the least beneficial pair-wise reallocations*

The most beneficial reallocations (RE^g ↓)

Reallocation across …	RE^g	TE^g	AE^g	$\overline{RE}$	$\overline{TE}$	$\overline{AE}$	RRE^g	TRE^g	ARE^g
1 and 5	1.8008	1.8004	1.0002	1.4999	1.4940	1.0040	1.2006	1.2051	0.9963
5 and 9	1.8096	1.8076	1.0011	1.4956	1.4879	1.0052	1.2099	1.2149	0.9959
5 and 10	1.8647	1.8645	1.0001	1.5382	1.5319	1.0041	1.2123	1.2171	0.9960
1 and 3	1.8919	1.8915	1.0002	1.5443	1.5382	1.0040	1.2251	1.2297	0.9962
3 and 9	1.8968	1.8951	1.0009	1.5365	1.5286	1.0052	1.2345	1.2397	0.9958

The least beneficial reallocations (RE^g ↑)

Reallocation across …	RE^g	TE^g	AE^g	$\overline{RE}$	$\overline{TE}$	$\overline{AE}$	RRE^g	TRE^g	ARE^g
3 and 8	2.1288	2.1283	1.0003	1.6253	1.6184	1.0043	1.3098	1.3150	0.9959
1 and 9	2.1263	2.1257	1.0003	1.6224	1.6180	1.0027	1.3106	1.3137	0.9976
2 and 6	2.1261	2.1256	1.0002	1.6302	1.6243	1.0036	1.3042	1.3086	0.9966
6 and 8	2.1260	2.1255	1.0003	1.6450	1.6380	1.0043	1.2925	1.2976	0.9960
2 and 4	2.1249	2.1243	1.0003	1.6242	1.6185	1.0035	1.3083	1.3126	0.9967

* the results for all 45 reallocations can be found in the Appendix 3.

From these results one my see that the reallocation across some units is quite beneficial for the group production, while it is not always so. Hence, it is possible to rank the pairs of units with respect to the gains from reallocating inputs across them. It is also evident that the reallocation, when it is done partially, is not necessarily beneficial for group revenue maximization. Although of total 45 possible reallocations 40 actually improved potential revenue efficiency of the group, the remaining 5 led to higher inefficiency (see the Appendix for the results of the reallocations across all 45 pairs of units). As we have already discussed above, reallocation from relatively larger and inefficient units to smaller and more efficient ones may lead to the decrease in potential revenue (in)efficiency, RE^g. However, it may be not the case if larger units are relatively more efficient than the smaller ones, and thus, reallocation may be not beneficial. This is what we observe in the estimations above. All of the least efficient reallocations involve relatively more efficient large units and less efficient small units (for the sizes and individual efficiency scores, refer to the Table 2). For these units, reallocative efficiency improves only marginally after reallocation, while structural efficiency worsens much, leading to the decrease in potential revenue efficiency.

To conclude this section, we summarize our findings. First, it has been demonstrated that the gains from inputs reallocation may be substantial. This fact motivates the practical application of the potential and reallocative efficiency measures, since if they are neglected, one may disregard potentially valuable information on how the performance of the group can be enhanced. Second, we have shown with an example that just reallocating inputs across all units is sufficient for the reallocation gains to be completely seized. Finally, we have provided an example of how one may evaluate the relative attractiveness of different units as candidates for the inputs reallocation.

C h a p t e r 5

LIMITATIONS AND FURTHER EXTENSIONS

Although our analysis involves the assumptions, which are traditionally made by the researchers in the efficiency and productivity field, such as the same access to technology and the regularity axioms, some of these assumptions are quite restrictive. From the one hand, these restrictive assumptions impose certain limits to this work. From the other hand, the problems raised by restrictive assumptions form the agenda for further research. Therefore, in this section we discuss the limitations of our work and connect them to possible extensions aimed at overcoming these limitations.

The major problem arises when one realizes that not all inputs may be reallocated between the units. For example, there may exist fixed costs, or unit-specific inputs, which can not be reallocated. Moreover, if the units are geographically disperse, it may be impossible or very costly to reallocate inputs across them. So, inputs may be reallocated, but only across some units. These two possibilities does not completely fit in our framework (yet), and more work is needed to make the measures of potential efficiency (and reallocative efficiency) to account for such possibilities.

Another question is whether it is really beneficial for the enterprise to reallocate inputs, and hence activities, across its units. First, there may exist economies from "horizontal" specialization across the units. Second, the enterprise may have a vertical structure, where outputs of some units are the inputs for the others. In such a case assumption of the same technology, and thus reallocation of inputs across the units may be irrelevant.

In our work, we assume that the output prices are the same for all units. Even in the multi-unit enterprise case, it seems to be not always reasonable assumption. The demand conditions may be different in different regions, as well as cost structure of different units, *etc*. The problem aggravates if one is to consider an industry, or the economy as a whole. Thus, it may be crucial to have reasonable information on output prices. Sometimes it is not a problem, but sometimes one needs to estimate these prices (e.g., one may estimate "shadow prices"), or proxy them with marginal costs or other proxies, which may be not so easy. But even if we have price information, we can not apply our theoretical results if the prices are different. This is so because of the assumption of the same prices, which was imposed to enable the aggregation of individual revenue functions.

The other restrictive assumption in this work is made rather implicitly, than explicitly. Specifically, it is assumed that total input endowment of the group of units is fixed, as well as the number of units. Although it may be not a great problem if one is interested in the point-in-time, static, analysis. However, it is often desirable to see how things behave in dynamics. For example, the question of interest may be how reallocative efficiency behaved over some period. Suppose, it has decreased. But it is impossible, at this stage of research, to answer whether it has happened due to more optimal allocation of existent inputs between the units or due to an increase of total input endowment of the enterprise, which in turn has led to more optimal allocation of resources. Or, alternatively, more optimal allocation was itself a result of changed number of units in the group. At this point, we leave these issues for further investigation.

Another issue, which is related to the previous one, is the possibility of altering the number of units within an enterprise. Specifically, it may be very useful, from the practical point of view, to compare gains from the reallocation of inputs to those from merging/separating the units, or/and from establishing new ones. Although some work has been done with regards to mergers (Bogetoft and Wang (1999)), the question of *measuring efficiency gains* from separating existent units and establishing new units is left unanswered at this point.

In the hypothetical example provided in the previous section, we say that, to achieve higher group revenue, even if all units are individually efficient, one should consider the possibility of reallocating the inputs across the units. However, the problem may appear when the decision about *how* such reallocation is to be made. As one possibility, the manager could perform cost-benefit analysis of inputs reallocation. But if the benefits are more or less clear (due to this research), the cost side of this process may be not very transparent. The costs may include transaction and transportation cost, miscellaneous labor costs, such as unemployment compensations and hiring costs, restructuring costs and others. So, the manager should take these costs into account when making a decision about inputs reallocation across the units.

From relatively more technical point of view, one may feel uncomfortable with the assumption of convex technology. Many real life production processes involve regular expansions of firm's capacity. The capacity may be being expanded over some period of time, and once it is eventually launched, the output may increase immediately. If the technology involves such a process, it is not convex anymore, and, at this stage, the results of this work can not be used in practice in such a case.

It is necessary to note that, in our hypothetical example we have explicitly assumed that we know "true" technology. Therefore, due to the convexity of the output set, our "average unit" was necessarily within the output set. However, the information about "true" technology is likely to be not available in real life. Thus, one would need to apply some sort of technique, such as DEA or Stochastic Frontier, to *estimate* the production frontier. The problem is that, if the number of observations is small, the DEA estimates will have substantial bias, and one is quite likely to find "average unit" out of the frontier. Thus, the estimate of group potential efficiency may happen to be less than unity, which is theoretically and practically meaningless. One possible way to deal with such a problem is to apply Stochastic Bootstrap technique to eliminate the DEA bias. The correctness of this suggestion should be evaluated in further research.

The reader may also ask why we did the analysis for the output oriented case and not for the input case. This is just because the output side seemed to be more intuitive for our purposes. Moreover, the work done may be "mirrored" in terms of the input measures. Though, the intuition will be somewhat different. As we have already mentioned in the Literature Review section, Färe *et al.* (1994) suggested a measure of *"efficiency gains due to product diversification"*. It is very similar to inputs reallocation, but now *outputs* are to be "reallocated" across the units, which may eventually lead to *cost economies* (see Färe *et al.* (1994), pp. 263-269). So, an interesting extension to this work would be to consider input oriented measures. Furthermore, one may turn to the directional distance functions in search of the measure of gains due to reallocation of both, inputs *and* outputs.

Chapter 6

CONCLUSIONS

In this paper, we have merged the prior the works of Li and Ng (1995) and Färe and Zelenyuk (2003), and developed necessary extensions, to provide theoretical grounds for the measurement of group efficiency when reallocation of inputs within the group is possible. In addition to the measure of group structural efficiency, which do not allow for inputs reallocation (Färe and Zelenyuk, 2003), we have developed a measure which allows for this possibility, group potential efficiency. We have shown that this measure can be decomposed into technical and allocative components, which may be of high practical use, since the source of overall inefficiency can be more clearly identified. Depending on the source of inefficiency, the implications for its removal may differ. Further, we have established the link between the two measures of group efficiency, potential and structural efficiencies. This link itself represents an efficiency measure, which is group reallocative efficiency. It shows how much group revenues may be increased, even if all its units are individually efficient, and represents gains due to the reallocation of inputs across the units. We then decomposed this measure to uncover the sources of such gains, which may appear either from unrealized group output, or from gains due to better match between individual and group-wise product mixes. Finally, we employed a hypothetical example to show how our measures work. Specifically, we have shown that, even if all units in our example were individually efficient, one may gain about 31% of extra revenues just reallocating inputs between these units. So, the gains from such a reallocation may be quite substantial, and they certainly should be measured if it is possible, in principle, to reallocate inputs within the group. This observation enforces the motivation to use the measures of potential and reallocative efficiency (in addition to the structural measures) in practice.

It should be stressed that the proposed approach to measuring the efficiency of a group may be useful in vast applications, because the issue of optimality of resources allocation is the central one in economics and can not be neglected. In general, the idea of reallocative efficiency can be applied whenever one considers the performance of some system and its constituent parts. As an illuminating example, we would refer to the work of Li and Ng (1999), where they in particular establish the result that Chinese state enterprises were very reallocatively inefficient. That is, by the means of reallocating activities across these enterprises, their aggregate output and revenue could be increased even holding individual inefficiencies constant. As a result, such a reallocation would lead to the enhancement of social welfare. This example illustrates the importance of allocation of productive resources in economy in general, and in the transition economies in particular.

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APPENDICES

A1. Description of the MalLab codes and data files

File name	Description
`genx.m`	generation of random individual inputs
`geny.m`	generation of individual technically efficient outputs
`genTE.m`	generation of random individual technical efficiencies
`genDataNew.m`	generation of initial individual inputs, outputs, technical, revenue and allocative efficiency measures
`genData.m`	is used in genDataNew.m, and for generation of initial individual outputs, revenue and allocative efficiency measures after reallocation
`effGroup.m`	calculation of group efficiency measures
`estRealc.m`	reallocation of inputs, reestimation of group efficiency measures (with effGroup.m) and storage of the results
`data0.mat`	initially stored data
`data.mat`	rewritable data, is used for storing temporary data (with reallocated inputs)
`results0.mat`	initially estimated individual and group efficiency measures
`results.mat`	rewritable file, used for storing reestimated individual and group efficiency measures after reallocation
`realcres10-10.mat`	resulted group efficiency measures after reallocation across all units
`realcres10-2.mat`	resulted group efficiency measures after pair-wise reallocations

A2. Randomly generated individual inputs (rounded)

Unit	Amount of input used for production of the first output	Amount of input used for production of the second output
1	980 390	1 020 706
2	17 234	40 169
3	15 820	19 847
4	70 641	63 910
5	1 002	877
6	100 131	102 798
7	134 007	130 062
8	8 217	7 462
9	560 141	1 116 454
10	87 356	726 324

A3. Resulted group measures after all possible pair-wise reallocations

Reallocation across …		RE^g	TE^g	AE^g	$\overline{RE}$	$\overline{TE}$	$\overline{AE}$	RRE^g	TRE^g	ARE^g
1	2	2.0026	2.0023	1.0001	1.6243	1.6190	1.0033	1.2329	1.2368	0.9969
1	3	1.8919	1.8915	1.0002	1.5443	1.5382	1.0040	1.2251	1.2297	0.9962
1	4	1.9689	1.9684	1.0003	1.5742	1.5678	1.0041	1.2507	1.2555	0.9962
1	5	1.8008	1.8004	1.0002	1.4999	1.4940	1.0040	1.2006	1.2051	0.9963
1	6	1.9684	1.9679	1.0003	1.5608	1.5544	1.0042	1.2611	1.2660	0.9961
1	7	2.0462	2.0457	1.0003	1.6135	1.6067	1.0042	1.2682	1.2732	0.9961
1	8	2.0228	2.0222	1.0003	1.6652	1.6585	1.0040	1.2148	1.2193	0.9963
1	9	2.1263	2.1257	1.0003	1.6224	1.6180	1.0027	1.3106	1.3137	0.9976
1	10	2.1141	2.1134	1.0003	1.6135	1.6068	1.0041	1.3103	1.3152	0.9962
2	3	2.1199	2.1193	1.0003	1.6176	1.6108	1.0042	1.3105	1.3156	0.9961
2	4	2.1249	2.1243	1.0003	1.6242	1.6185	1.0035	1.3083	1.3126	0.9967
2	5	2.0799	2.0790	1.0004	1.6028	1.5955	1.0046	1.2977	1.3031	0.9959
2	6	2.1261	2.1256	1.0002	1.6302	1.6243	1.0036	1.3042	1.3086	0.9967
2	7	2.1097	2.1093	1.0002	1.6222	1.6165	1.0035	1.3005	1.3048	0.9967
2	8	2.1226	2.1219	1.0003	1.6236	1.6168	1.0042	1.3073	1.3124	0.9961
2	9	1.9959	1.9949	1.0005	1.6067	1.5993	1.0047	1.2422	1.2474	0.9959
2	10	2.0580	2.0578	1.0001	1.6536	1.6482	1.0033	1.2445	1.2485	0.9968
3	4	2.1083	2.1078	1.0002	1.6155	1.6087	1.0042	1.3051	1.3103	0.9960
3	5	2.0994	2.0988	1.0003	1.6125	1.6056	1.0043	1.3020	1.3072	0.9960
3	6	2.1003	2.0998	1.0002	1.6158	1.6090	1.0042	1.2999	1.3051	0.9960
3	7	2.0758	2.0753	1.0002	1.6023	1.5956	1.0042	1.2955	1.3007	0.9960
3	8	2.1288	2.1283	1.0003	1.6253	1.6184	1.0043	1.3098	1.3150	0.9960
3	9	1.8968	1.8951	1.0009	1.5365	1.5286	1.0052	1.2345	1.2397	0.9958
3	10	1.9554	1.9552	1.0001	1.5810	1.5745	1.0041	1.2368	1.2418	0.9960
4	5	2.0558	2.0553	1.0002	1.5977	1.5909	1.0043	1.2867	1.2919	0.9960
4	6	2.1245	2.1239	1.0003	1.6215	1.6145	1.0043	1.3102	1.3155	0.9960
4	7	2.1139	2.1133	1.0003	1.6155	1.6086	1.0043	1.3085	1.3138	0.9960
4	8	2.1247	2.1241	1.0003	1.6358	1.6288	1.0043	1.2988	1.3041	0.9960
4	9	1.9701	1.9683	1.0009	1.5640	1.5566	1.0048	1.2596	1.2645	0.9962
4	10	2.0291	2.0287	1.0002	1.6080	1.6013	1.0042	1.2619	1.2669	0.9960
5	6	2.0414	2.0409	1.0003	1.5955	1.5888	1.0042	1.2795	1.2846	0.9961
5	7	2.0108	2.0103	1.0002	1.5783	1.5717	1.0042	1.2741	1.2791	0.9961
5	8	2.1064	2.1058	1.0003	1.6116	1.6047	1.0043	1.3070	1.3123	0.9960
5	9	1.8096	1.8076	1.0011	1.4956	1.4879	1.0052	1.2099	1.2149	0.9959
5	10	1.8647	1.8645	1.0001	1.5382	1.5319	1.0041	1.2123	1.2171	0.9960
6	7	2.1200	2.1194	1.0003	1.6174	1.6104	1.0043	1.3107	1.3160	0.9960
6	8	2.1260	2.1255	1.0003	1.6450	1.6380	1.0043	1.2925	1.2976	0.9960
6	9	1.9729	1.9710	1.0010	1.5540	1.5466	1.0048	1.2696	1.2744	0.9962
6	10	2.0290	2.0287	1.0002	1.5956	1.5889	1.0042	1.2717	1.2768	0.9959
7	8	2.1065	2.1059	1.0003	1.6362	1.6293	1.0042	1.2874	1.2926	0.9960
7	9	2.0404	2.0390	1.0007	1.5987	1.5915	1.0045	1.2763	1.2812	0.9962

7	10	2.0968	2.0963	1.0002	1.6403	1.6334	1.0043	1.2783	1.2834	0.9960
8	9	2.0103	2.0091	1.0006	1.6422	1.6337	1.0052	1.2242	1.2298	0.9954
8	10	2.0756	2.0751	1.0002	1.6922	1.6853	1.0041	1.2265	1.2313	0.9961
9	10	2.1184	2.1171	1.0006	1.6157	1.6128	1.0018	1.3112	1.3127	0.9988